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Superposable Fluid Motions

J. A. Strang

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In the theory of fluid motions the equations of motion are not linear. We are therefore frequently compelled to resort to statistical or approximate methods of solution, the latter depending on the assumption that certain terms, usually the quadratic terms, are small compared with those retained, so that the solutions obtained are valid only when the motion is slow; or else it is tacitly assumed that two or more distinct motions are linearly superposable. In some cases both assumptions are made.

The former is the basic assumption in the theory of the boundary layer and the various approximations to the wake behind a solid obstacle. The latter is used openly, with perhaps the perfunctory statement «if the motions can be superposed» in Rankine's method of constructing the stream lines of the resultant of two two-dimensional motions when those of the components are known: and it is used tacitly again and again, as for instance in Lamb, Hydrodynamics, 6 th. edition, p. 644, when in the case of two-dimensional motion it is assumed that the solution is of the form

\[ u = -\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}, \quad v = -\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}. \]

This assumes that the motion whose stream function is \( \psi \) is linearly superposable on the irrotational motion whose velocity
potential is \( \varphi \). It is true that the superposability is in this instance secured by choosing \( \varphi \) and \( \psi \) in such a way that the equations of motion are satisfied, and that the quadratic terms having been in fact discarded altogether the choice is possible. But this does not alter the fact that the assumption of linear superposability has been made.

The same assumption occurs in the treatment of three dimensional motion on p. 209, where there is no suggestion of discarding quadratic terms. The assumption here is that we may superpose the velocity at any point due to a vortex system on that which is due to sources. The operative sentence is (p. 209, middle).

"The complete solution of our problem is obtained by superposition..." The italics are not Lamb's.

Again, consider the proof in paragraph 147, p. 207 from this point of view. It clearly assumes that if \( U_1 \) and \( U_2 \) are solutions of the equations of motion \( U_1 - U_2 \) is also a solution. That is precisely the point. The theorem will be true subject to the stated conditions if, but only if, the two solutions are linearly superposable: and pairs of solutions are not in general linearly superposable whether the other conditions of the theorem are fulfilled or not.

The idea of this paper, that two solutions may or may not be linearly superposable in the case of the hydrodynamical equations of motion, is capable of much wider application to any system of nonlinear equations, differential or otherwise. The corresponding inquiry would be whether there exist sets of solutions which are linearly additive; what restrictions are necessary and sufficient to secure the additive property; what are the special properties of such sets; whether all solutions belong to at least one additive set or not; and how to determine the members of the set which are additive to a given solution when such members exist.

It will be shown in what follows that in the case of fluid motions of a viscous incompressible fluid there exist such additive sets; and that in many cases to each fluid motion there corresponds an infinite class with which it forms an additive set; the class is defined by the superposability condition.
There is a large class of motions which are self superposable, including, but more extensive than, the class of all irrotational motions. These furnish a generalisation of Bernoulli's equation. It is shown that we can always construct a class of rotational motions which are superposable on a given irrotational motion, and that provided certain simple conditions are satisfied we can construct an irrotational motion which is superposable on a given rotational motion. The properties of plane motions are studied in some detail, and in passing we obtain a generalisation, as well as a wider interpretation, of a condition originally due to Stokes, given by him and in current text books as the condition for the existence of steady two-dimensional motion of a non-viscous homogeneous incompressible fluid. A number of hitherto unrelated solutions obtained for the motion of a viscous fluid are shown to be in fact self superposable motions; and certain three-dimensional groups of superposable motions are indicated.

2. The superposability condition

Let \( U_1 = (u_1, v_1, w_1) \) and \( U_2 = (u_2, v_2, w_2) \) be two solutions of the equations of motion of a viscous incompressible fluid

\[
\frac{DU}{Dt} = F - \frac{1}{\rho} \nabla p + \mu \cdot \nabla^2 U, \quad \left( \frac{D}{Dt} = \frac{\partial}{\partial t} + U \cdot \nabla \right)
\]

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (2.1)
\]

\[
\nabla U = 0, \quad (2.2)
\]

corresponding to given external forces, initial conditions, boundaries and boundary conditions, not necessarily the same in both cases. Let the pressures and external forces be \( p_1, F_1 \) and \( p_2, F_2 \) respectively, and let

\[
\frac{1}{\rho} \nabla p = \nabla P.
\]

The two solutions are said to be superposable if a pressure-density function \( P_1 + P_2 + \pi \) can be so determined that the
fluid moves with velocity $U_1 + U_2$ under external forces $F_1 + F_2$ and the pressure-density function $P_1 + P_2 + \pi$.

This is not the only possible definition. We might require that $(U_1, P_1), (U_2, P_2)$ and $(U_1 + U_2, P_1 + P_2 + \pi)$ should satisfy the equations of motion under the same force system $F$. It will however appear presently that if the force system is conservative precisely the same results are obtained provided that we replace $\pi$ by $\pi - \Omega$, where $\Omega$ is the force potential ($F = - \nabla \Omega$); and the condition of superposability is the same whichever definition is adopted.

The condition as might be expected depends only on the quadratic terms in the equations of motion. If for instance the axes are rotating about the $z$ axis, the equations of motion will contain additional terms due to the rotation. But provided that the angular velocities of the axes are independent of the time, and their squares and products are negligible, the additional terms are linear in the velocities and therefore do not affect conclusions regarding superposability. The condition remains unaltered.

For superposability we require

$$\frac{DU_1}{Dt} = F_1 - \nabla P_1 + \nu \cdot \nabla^2 U_1,$$

$$\frac{DU_2}{Dt} = F_2 - \nabla P_2 + \nu \cdot \nabla^2 U_2,$$

$$\frac{D}{Dt} (U_1 + U_2) = (F_1 + F_2) - \nabla (P_1 + P_2 + \pi) + \nu \cdot \nabla^2 (U_1 + U_2).$$

The elimination of $P_1$ and $P_2$ gives

$$- \nabla \pi = \nabla (U_1 \cdot U_2) - U_1 \times (\nabla \times U_2) - U_2 \times (\nabla \times U_1) \quad (2.3)$$

and the necessary and sufficient condition of superposability is that this should determine $\pi$, i.e. there must exist a scalar function $\gamma$ such that

$$U_1 \times (\nabla \times U_2) + U_2 \times (\nabla \times U_1) = \nabla \gamma. \quad (2.4)$$

The same condition is obtained from the second definition of superposability. If in the first two equations of motion $F_1 =$
$F_2 = F$, and if in the third equation $F_1 + F_2$ is replaced by $F$, the left hand side of (2.3) will be

$$-\nabla \pi - F = -\nabla (\pi - \Omega)$$

if $F$ is conservative, and (2.4) is unchanged.

It is evident that

$$\nabla (U_1 + U_2) = 0 \text{ if } \nabla U_1 = 0 \text{ and } \nabla U_2 = 0,$$

so that (2.2) requires no further attention for the present.

When $U_1 = U_2 = U$ we obtain the condition for self superposability,

$$U \times (\nabla \times U) = \nabla \chi,$$  \hspace{1cm} (2.5)

where as before $\chi$ means any scalar.

This condition means that the space rate of change of velocity is derivable from a potential, for

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{1}{2} \frac{\partial U^2}{\partial x} - v \zeta + w \eta$$

$$= \frac{\partial}{\partial x} \left( \frac{1}{2} U^2 - \chi \right),$$  \hspace{1cm} (2.6)

and similarly for the other components.

It also has a simple geometrical meaning. It states that there exists a family of surfaces $\chi =$ constant containing both the stream line and the vortex line at any point of the fluid; or else, if $\nabla \chi = 0$, either

(i) the vortex lines do not exist because the motion is irrotational, i.e.

$$\nabla \times U = 0;$$

or (ii) the vortex lines are at every point codirectional with the stream lines, i.e.

$$\nabla \times U = \lambda U,$$

where $\lambda$ is a scalar.

We can apply the superposability condition to investigate the validity of the assumption that
is an adequate representation of any two dimensional motion. Let us take

\[ u = -\frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial y}, \]
\[ v = -\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x}, \]

so that

\[ \zeta_1 = \eta_1 = \zeta_1 = 0, \text{ and } \zeta_2 = \eta_2 = 0, \quad \zeta_2 = \nabla_1 \psi, \quad \left( \nabla_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \]

The superposability condition then requires that

\[ \chi_x = -\zeta_2 \frac{\partial \varphi}{\partial y} = \zeta_2 \frac{\partial \theta}{\partial x}, \]
\[ \chi_y = \zeta_2 \frac{\partial \varphi}{\partial x} - \zeta_2 \frac{\partial \theta}{\partial y}, \]
\[ \chi_z = 0, \]

where \( \theta \) is the harmonic conjugate of \( \varphi \); and consistency requires that \( \zeta \) be a function of \( \theta \), i.e., the superposability condition is that

\[ \nabla_1 \psi = f(\theta), \]

where \( f(\theta) \) is an arbitrary function. When this condition is satisfied

\[ \chi = \int f(\theta) \, d\theta. \]

3. There are a number of simple consequences. For brevity we shall in what follows frequently use s. and ss. to denote superposable (or superposability) and self superposable respec-
tively, and the statement that $U_2$ is superposable on $U_1$ will be written $U_2$ s. $U_1$.

(a) If $U_1$ and $U_2$ are s. and ss., $U_1 \pm U_2$ are ss.

(It is not sufficient that both be ss.)

(b) If $U_1, U_2$ and $U_1 \pm U_2$ are ss., $U_2$ s. $U_1$.

(c) If $U_1$ s. $U_3$ and $U_2$ s. $U_3$, then $U_1 \pm U_2$ s. $U_3$.

(d) All irrotational motions are ss. and also s. on each other.

For in (2.4) and (2.5)

$$\nabla \times U_1 = 0 \quad \text{and} \quad \nabla \times U_2 = 0,$$

and so $\nabla \chi = 0$.

This property is one of two reasons why irrotational motions are of such importance in hydrodynamics. The other is the existence of a velocity potential. The two properties are not coextensive. The class of motions which are ss. and also s. on each other includes for instance all motions for which

$$\nabla \times U = \lambda U$$

and $\lambda$ has a common value, while irrotational motions are only a part of this class, corresponding to $\lambda = 0$. If $\lambda \neq 0$ the stream lines coincide with the vortex lines.

(e) If $\nabla \times U_1 = \lambda_1 U_1$ and $\nabla \times U_2 = \lambda_2 U_2$,

i.e. if in both motions the vortex lines coincide with the stream lines, the s. condition becomes

$$(\lambda_2 - \lambda_1) (U_1 \times U_2) = \nabla \chi,$$

which is certainly true if $\lambda_1 = \lambda_2$ and $\nabla \chi = 0$, i.e. if both $U_1$ and $U_2$ belong to the same class mentioned in (d).

(f) If $\nabla \times U_2 = \mu_2 U_1$ and $\nabla \times U_1 = \mu_1 U_2$

where $\mu_1$ and $\mu_2$ are any scalars, the left hand side of (2.4) is identically zero, we can take $\nabla \chi = 0$, and $U_2$ s. $U_1$. The stream lines of either motion coincide with the vortex lines of the other.

(g) More generally, if

$$\nabla \times U_1 = \lambda_1 U_1 + \mu_1 U_2$$

and $\nabla \times U_2 = \lambda_2 U_2 + \mu_2 U_1$,

the s. condition becomes $$(\lambda_2 - \lambda_1) (U_1 \times U_2) = \nabla \chi.$$
This condition is satisfied if

(i) \( \lambda_1 \neq \lambda_2, \ \nabla \chi \neq 0 \), and there exists a family of surfaces \( \chi = \) constant containing both families of stream lines:

or (ii) \( \lambda_1 = \lambda_2, \ \nabla \chi = 0 \). This includes both (e) and (f):

or (iii) \( U_1 \times U_2 = 0, \ \nabla \chi = 0 \).

This implies \( U_2 = k U_1 \), i.e. both motions have the same stream lines. But it also implies

\[
\nabla \times U_1 = (\lambda_1 + k \mu_1) U_1 \text{ and } \nabla \times (k U_1) = (\mu_2 + k \lambda_2) U_1,
\]

and combining these two we get

\[
(\nabla k) \times U_1 = [\mu_2 + (\lambda_2 - \lambda_1) k - \mu_1 k^2] U_1.
\]

Since the non-zero vector \( U_1 \) cannot be perpendicular to itself both sides must vanish. Hence \( k \) must be a root of the equation

\[
\mu_2 + (\lambda_2 - \lambda_1) k - \mu_1 k^2 = 0,
\]

and either \( \nabla k = 0 \), i.e. \( k \) is a constant, or else

\( U_1 = \theta \cdot \nabla k \), and of course \( U_2 = k \theta \cdot \nabla k \),

where \( \theta \) is a scalar. In the latter case there exists a family of surfaces orthogonal to the stream lines of \( U_1 \) and \( U_2 \), so that

\[ U_1 \cdot (\nabla \times U_1) = 0. \]

(h) In all integrable cases, i.e. all cases in which \( U_2 \) s. \( U_1 \),

\[ U_1 \cdot U_2 - \chi + \pi = \text{a constant}, \]

which may be written

\[ U_1 \cdot U_2 + \pi = \chi, \]

since the constant may be included in \( \chi \). The function \( \chi \) may contain \( t \) also, since the integration relates only to \( x, y, z \).

If we adopt the second definition the result will be

\[ U_1 \cdot U_2 + \pi - \Omega = \chi, \]

and when \( U_1 = U_2 \),

\[ U_2 + \pi - \Omega = \chi. \]
4. The construction of motions superposable on a given $U_1$.

The preceding paragraph indicates clearly the possibility of constructing motions $U_2$ s. $U_1$ under certain conditions to be satisfied by $U_1$. But it is of particular interest to know whether such motions can be constructed when $U_1$ is irrotational, and whether we can construct irrotational motions $U_2$ s. $U_1$ when $U_1$ is rotational.

A. If $U_1 = -\nabla \varphi$ is the velocity of any irrotational motion we can always obtain a class of rotational $U_2$ s. $U_1$.

The s. condition

$$U_1 \times (\nabla \times U_2) + U_2 \times (\nabla \times U_1) = \nabla \chi$$

becomes

$$\mathcal{W} \times \nabla \varphi = \nabla \chi,$$

where $\mathcal{W} = \nabla \times U_2$.

The condition of solubility

$$\nabla \varphi \cdot \nabla \chi = 0$$

implies that when $\varphi$ is given the vector $\nabla \chi$ is perpendicular to $\nabla \varphi$, and analytically that $\chi$ involves an arbitrary function. When this condition is satisfied the equation

$$\mathcal{W} \times \nabla \varphi = \nabla \chi$$

requires only

$$|\mathcal{W}| \cdot |\nabla \varphi| \cdot \sin \theta = |\nabla \chi|,$$

where the angle $\theta$ is measured from the direction of $\mathcal{W}$ to that of $\nabla \varphi$, so that $\mathcal{W}$ depends on two variables $\chi$ and $\theta$, of which $\theta$ is entirely arbitrary, while $\chi$ involves an arbitrary function. In addition we must have $\nabla \mathcal{W} = 0$. This involves at most a relation between the two arbitrary elements.

When $\mathcal{W}$ has been determined we have to choose $U_2$ so that

$$\nabla \times U_2 = \mathcal{W}, \quad \nabla U_2 = 0.$$  

The condition for solubility, namely $\nabla \mathcal{W} = 0$, is already satisfied.

If now we write $U_2 = \nabla \times V$

the second condition is an identity, while the first becomes
\[ \nabla \times (\nabla \times V) = W, \]
i.e.
\[ \nabla^2 V = -W, \]
of which the solution is known. Hence the general solution is
\[ U_3 = -\nabla H + \nabla \times V, \]
where \( V \) has the value obtained above, and \( H \) is any harmonic function.

These solutions are not in general ss.

Two particular cases are of interest.

(i) When \( \nabla \chi = 0 \) a solution of the first equation \( W \times \nabla \varphi = 0 \) is given by
\[ W = \lambda \cdot \nabla \varphi, \]
where \( \lambda \) is a constant, and we can take
\[ u = -H_x + \lambda \varphi_y, \]
\[ v = -H_y - \lambda \varphi_x, \]
\[ w = -H_z, \]
where \( \varphi = \varphi_z \), \( \nabla \varphi = 0 \), and \( \nabla^2 H = 0 \); for these satisfy all the conditions.

(ii) If the irrotational motion does not depend on \( z \), i.e. \( \varphi_z = 0 \), let \( \theta \) be the harmonic conjugate of \( \varphi \).

The condition \( \nabla \varphi \cdot \nabla \chi = 0 \) is satisfied by
\[ \chi_x = -A \cdot \varphi_y = A \cdot \theta_x, \]
and \( \chi_y = A \cdot \varphi_x = A \cdot \theta_y, \)
while the consistency condition is satisfied if \( A = f(\theta) \), where \( f(\theta) \) is an arbitrary function. This is the result already stated at the end of paragraph 2.

The equation
\[ W \times \nabla \varphi = \nabla \chi, \]
leads at once to \( \zeta = B \cdot \varphi_z, \eta = B \cdot \varphi_y, \zeta = f(\theta), \)
and in order to satisfy \( \nabla W = 0 \) we must have \( B = g(\theta) \), where \( g(\theta) \) is arbitrary.
Superposable Fluid Motions

The final solution is therefore obtained by solving the equations
\[ w_y - u_x = g(\theta) \cdot \theta_y, \]
\[ u_x - w_x = -g(\theta) \cdot \theta_x, \]
\[ v_x - u_y = f(\theta), \]
\[ u_x + v_y + w_z = 0. \]

In particular if \( u \) and \( v \) do not depend on \( z \) we can take
\[ w = \int g(\theta) \cdot d\theta \]
and \[ v_x - u_y = f(\theta), \]
and since in this case we have also
\[ u_x + v_y = 0, \]
we can take
\[ u = -\psi_y, \]
\[ v = \psi_x, \]
provided that \( \nabla^2 \psi = f(\theta) \), and the equations of motion are satisfied.

B. If \( U_1 \) is rotational we can always construct an irrotational \( U_2 = -\nabla \phi \) which is s, \( U_1 \), provided that \( U_1 \) is harmonic, or what is the same thing, either
\[ \nabla \times (\nabla \times U_1) = 0 \quad \text{or} \quad \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial z}, \quad \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial z}. \]

The s. condition is again
\[ W_1 \times \nabla \phi = \nabla \chi, \]
and is satisfied by
\[ \nabla \phi = \lambda \cdot W_1, \quad \nabla \chi = 0, \]
where \( \lambda \) is constant, provided that the equations
\[ \phi_x = \lambda \xi, \quad \phi_y = \lambda \eta, \quad \phi_z = \lambda \zeta \]
are consistent and make
\[ \nabla^2 \phi = 0. \]
Consistency is secured by the condition $\nabla \times (\nabla \times U) = 0$, and evidently
$$\nabla^2 \varphi = \lambda \cdot \nabla W_1 = 0,$$
so that we may take
$$\varphi = \lambda \int (\xi \cdot dx + \eta \cdot dy + \zeta \cdot dz).$$

If $\nabla \chi \neq 0$ a necessary condition is $W_1 \cdot \nabla \chi = 0$. This being true we may take
$$\nabla \varphi = \lambda \cdot W_1 + V,$$
where $V$ is any solution of
$$W_1 \times V = \nabla \chi$$
satisfying the same conditions as $W_1$, namely
$$\nabla V = 0 \text{ and } \nabla \times V = 0.$$

5. Plane motions

When the stream lines are plane curves in parallel planes we can use the stream function $\psi$ with advantage to interpret the s. and ss. conditions. We assume that $w = 0$, but that $u$, $v$, and $\psi$ may depend on $z$. Dependence on $t$ is implicit throughout. If $u$, $v$ and $\psi$ are independent of $z$ we have the usual two dimensional motion.

The equation of continuity
$$\nabla U = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
is satisfied by
$$u = -\psi_y, \quad v = \psi_x,$$
and
$$\xi = w_y - u_z = -\psi_{yz},$$
$$\eta = u_z - w_x = -\psi_{xz},$$
$$\zeta = v_x - u_y = \nabla_1^2 \psi.$$

The stream function must be a solution of the equation
\[
\left( \frac{\partial}{\partial t} - \nu \nabla^2 \right) \nabla_1^2 \psi = 0
\]

in order that the equation of motion may be satisfied.

The s. condition requires the existence of a function \( \chi \) such that

\[
\begin{align*}
\chi_x &= \psi_{1x} \zeta_2 + \psi_{2x} \zeta_1 \\
\chi_y &= \psi_{1y} \zeta_2 + \psi_{2y} \zeta_1 \\
\chi_z &= \frac{\partial}{\partial z} (\psi_{1z} \psi_{2x} + \psi_{1y} \psi_{2y})
\end{align*}
\] (5.1)

From the third equation

\[
\chi = \psi_{1x} \psi_{2z} + \psi_{1y} \psi_{2y} + g(x, y)
\] (5.2)

where \( g(x, y) \) does not contain \( z \).

The first and second equations require that

\[
\nabla_1^2 \psi_2 = \zeta_2 = \frac{\partial f}{\partial \psi_1}, \quad \nabla_1^2 \psi_1 = \zeta_1 = \frac{\partial f}{\partial \psi_2},
\] (5.3)

where \( f(\psi_1, \psi_2) \) may be any function of \( \psi_1, \psi_2, z \). These two equations are then equivalent to

\[
\chi = f(\psi_1, \psi_2);
\] (5.4)

that is, \( \psi_2 \) must in general be expressible as a function of \( \psi_1 \) and \( \chi \), say

\[
\psi_2 = F(\chi, \psi_1),
\]

and it only remains to satisfy (5.2), i.e.

\[
\left( \psi_{1x}^2 + \psi_{1y}^2 \right) \frac{\partial F}{\partial \psi_1} + \left( \psi_{1x} \chi_x + \psi_{1y} \chi_y \right) \frac{\partial F}{\partial \chi} = \chi - g(x, y)
\] (5.5)

as the final requirement for the determination of \( \psi_2 \) when \( \psi_1 \) is given, apart from the equation of motion itself.

In general (5.5) is integrable, i.e. a motion \( U_0 \) s. \( U_1 \) exists, only if the ratios of the coefficients

\[
\psi_{1x}^2 + \psi_{1y}^2, \quad \psi_{1x} \chi_x + \psi_{1y} \chi_y, \quad \chi - g(x, y)
\] (5.6)
can be expressed as functions of $\psi_1$, $\chi$ and $z$; and we have at our disposal the functions $\chi$ and $g(x, y)$.

Two particular cases are of interest.

If in any plane $z = \text{constant}$ $U_1$ and $U_2$ are both irrotational

$$\zeta_1 = \nabla^2 \psi_1 = 0, \quad \zeta_2 = \nabla^2 \psi_2 = 0, \quad \chi = \chi_y = 0,$$

and it must be possible to choose $g(x, y)$ so that

$$\chi = \psi_{1x} \psi_{2x} + \psi_{1y} \psi_{2y} + g(x, y)$$

is a function of $z$ only. This is evidently true when $\psi_1$ and $\psi_2$ depend only on $x$ and $y$, i.e. when $U_1$ and $U_2$ are two dimensional irrotational motions. In that case $f(\psi_1, \psi_2) = 0$.

If $U_1$ is irrotational and $\nabla^2 \psi_1 = 0$ it follows from (5.1) that $\psi_2$ and $\chi$ are functions of $\psi_1$ and $z$ only, i.e. in every plane $z = \text{const.}$ the stream lines of $U_2$ coincide with those of $U_1$; but the relation between $\psi_1$ and $\psi_2$ will in general contain $z$ also. If $\psi_2 = F(\psi_1)$ the equation (5.2) becomes

$$\chi = (\psi_{1x} + \psi_{1y}) \frac{dF}{d\psi_1} + g(x, y) \quad (5.7)$$

which, since $\chi$ is a function of $\psi_1$, can be made integrable by choosing $\psi_1$ so that

$$\psi_{1x} + \psi_{1y} \text{ is a function of } \psi_1.$$

This condition is compatible with $\nabla^2 \psi_1 = 0$.

The ss. conditions for plane motion are obtained at once from (5.1) by making $\psi_1 = \psi_2 = \psi$, when we obtain

and

$$\zeta = \nabla^2 \psi = F(\psi) \quad (5.8)$$

$$\chi = \psi_{x} + \psi_{y} + g(x, y) = \int F(\psi) \, d\psi. \quad (5.9)$$

When $\psi$ does not depend on $z$, i.e. in two dimensional motion, (5.9) is automatically satisfied.

The equation (5.8), of which equations (5.3) are a generalisation, was given by Stokes for the case of two dimensional
motion as a condition for the possibility of steady motion of a homogeneous incompressible nonviscous fluid, and is so given in textbooks today. Its true meaning is wider. It is the condition that the motion be ss., and is applicable with (5·9) to any plane motion, steady or otherwise, of a viscous or nonviscous homogeneous incompressible fluid.

Similarly the equations given by Lamb, p. 244 (3), are equivalent to equation (2·5) of this paper, but in Lamb they are stated to be the conditions for a possible state of steady motion in three dimensions, which is true only of a nonviscous fluid. The are merely the conditions that the motion be ss., and the textbook use of them merely amounts to the statement that a steady motion of a nonviscous homogeneous incompressible fluid must be ss. But this follows immediately from the equation of motion, which in that case, since it may be written

$$\frac{\partial U}{\partial t} - v \cdot \nabla^2 U = \nabla (\lambda - \chi^l)$$

reduces to

$$0 = \nabla (\lambda - \chi^l)$$

and so establishes the existence of \( \chi \).

The other remark suggested by the equation (5·8) is the obvious one that the sum of two ss. solutions is again a solution if and only if

$$F(\psi) = k \psi,$$

where \( k \) is a constant so far as \( x \) and \( y \) are concerned, and is the same for both solutions. If \( k = 0 \) the solutions are irrotational; if \( k \neq 0 \) they are rotational. In these cases, and in these only, Rankine's construction of the stream lines of the resultant of two plane motions is applicable.

The reason for the limitation may be seen otherwise. It is to be found in the s. conditions (5·3). If \( U_1 \) and \( U_2 \) are ss., and if \( U_1 + U_2 \) also is to be ss., then we must have \( U_2 \) s. \( U_1 \). Now since \( U_1 \) and \( U_2 \) are ss.,

$$\zeta_1 = F_1(\psi_1) \text{ and } \zeta_2 = F_2(\psi_2),$$

and combining these with the equations (5·3) to ensure the s. property
\[ \frac{\partial f}{\partial \psi_2} = F_1(\psi_1) \text{ and } \frac{\partial f}{\partial \psi_1} = F_2(\psi_2), \]

which are true only if

\[ f(\psi_1, \psi_2) = k \psi_1 \psi_2, \quad F_1 = k \psi_1, \quad F_2 = k \psi_2. \]

6. **Ss. motions**

(1) If \( U \) is ss. the function \( \chi - \chi^t \) is a harmonic function, where

\[ U \times (\nabla \times U) = \nabla \chi, \text{ and } \chi^t = \int \frac{dp}{p} + \frac{1}{2} U^2 + \Omega. \]

The equation of motion may be written

\[ \left( \frac{\partial}{\partial t} - \nu \cdot \nabla^2 \right) U = \nabla(\chi - \chi^t), \quad (6.1) \]

and on using the operator \( \nabla \cdot \) it follows that

\[ \left( \frac{\partial}{\partial t} - \nu \cdot \nabla^2 \right) \nabla U = \nabla^2 (\chi - \chi^t), \]

i.e.

\[ \nabla^2 (\chi - \chi^t) = 0 \quad (6.2) \]

so that \( \chi - \chi^t \) is a harmonic function of \( x, y, z \).

This generalises Bernoulli's equation to all ss. motions.

If

\[ \left( \frac{\partial}{\partial t} - \nu \cdot \nabla^2 \right) U = 0 \]

it follows from (6.1) that

\[ \chi - \chi^t = F(t) \quad (6.3) \]

which reduces to the usual form of Bernoulli's equation if \( U \) is steady and irrotational, for then \( \chi = 0 \).

(2) Another immediate consequence is obtained by using the operator \( \nabla \times \), which furnishes

\[ \left( \frac{\partial}{\partial t} - \nu \cdot \nabla^2 \right) W = 0, \quad (W = \nabla \times U) \quad (6.4) \]
i.e. in ss. motion each component of vorticity is propagated independently in accordance with the simple diffusion equation.

The equation (6.4) may also be regarded as the ss. condition, since it is the condition of integrability of (6.1), i.e. the condition for the existence of a function \( \chi \).

Contrast the simplicity of this result with the equations (8) given in Lamb, p. 578,

\[
\frac{D\xi}{Dt} = \xi \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial y} + \zeta \frac{\partial u}{\partial z} + \nu \cdot \nabla^2 \xi,
\]

\[
\frac{D\eta}{Dt} = \xi \frac{\partial v}{\partial x} + \eta \frac{\partial v}{\partial y} + \zeta \frac{\partial v}{\partial z} + \nu \cdot \nabla^2 \eta,
\]

\[
\frac{D\zeta}{Dt} = \xi \frac{\partial w}{\partial x} + \eta \frac{\partial w}{\partial y} + \zeta \frac{\partial w}{\partial z} + \nu \cdot \nabla^2 \zeta.
\]

In particular it follows that in ss. motion of a nonviscous fluid the vorticity cannot increase or diminish with the time; if in ss. motion of a viscous fluid the vorticity is independent of the time it is harmonic; and if the vorticity is harmonic it is independent of the time.

(3) As an instance of the use which may be made of these ideas let us take the case of a uniform cylindrical tube of internal radius \( a \) and inquire whether ss. motion of a viscous fluid can be found of type

\[
u = f(r) T_1,
\]

\[
u = z \cdot g(r) T_2,
\]

\[
u = -y \cdot g(r) T_2,
\]

where \( r^2 = y^2 + z^2 \), \( f(r) \) and \( g(r) \) are functions to be determined, and \( T_1, T_2 \) are functions of \( t \) only. This amounts to a velocity parallel to the axis and an angular velocity about it both of which may depend on \( r \).

Calculating \( \xi, \eta, \zeta \) and then \( \chi \) we find
\[ \xi = -(2g + r g') T_1, \]
\[ \eta = \frac{z}{r} f' T_1, \]
\[ \zeta = -\frac{y}{r} f' T_1. \]

\[ \chi_x = v_x^2 - w_y = 0, \]
\[ \chi_y = w_x^2 - u_z = \frac{\partial}{\partial y} \left[ \frac{1}{2} f^2 T_1 + T_2 \int r g (rg' + 2g) dr \right], \]
\[ \chi_z = u_y - v_x = \frac{\partial}{\partial z} \left[ \frac{1}{2} f^2 T_1 + T_2 \int r g (rg' + 2g) dr \right], \]

so that the motion is ss. for any choice of \( f, g, T_1 \) and \( T_2 \); \( \nabla U = 0 \); and

\[ \chi = \frac{1}{2} f^2 T_1^2 + T_2^2 \int r g (rg' + 2g) dr. \]

It only remains therefore to satisfy the equations of motion and the boundary and initial conditions.

The equations of motion require that

\[ \frac{\partial}{\partial x} (\chi - \chi') = f T_1' - v T_1 \left( f'' + \frac{1}{2} f' \right), \]
\[ \frac{\partial}{\partial y} (\chi - \chi') = zg T_2' - v z T_2 \left( g'' + \frac{3}{r} g' \right) = zA \text{ say}, \]
\[ \frac{\partial}{\partial z} (\chi - \chi') = -yg T_2' + v y T_2 \left( g'' + \frac{3}{r} g' \right) = -yA. \]

Since each of the three right hand sides is a function of \( y \) and \( z \) only, the first must be a function of \( t \) at most, say \( F(t) \), and the consistency of the two latter equations requires that

\[ 2A + rA_{r} = 0, \]

so that \( r^2A \) is at most a function of \( t \), say \( G(t) \). The most general form of \( \chi - \chi' \) is therefore
\[ \chi - \chi^1 = x F(t) + G(t) \text{ arc tan} \left( \frac{y}{z} \right) + H(t). \]

If the pressure and the force potential however are single valued \( G(t) = 0 \) identically. We assume this.

The axial velocity therefore depends on the equation

\[ fT_1' - \nabla T_1 \left( f' + \frac{1}{r} f' \right) = F(t), \]

while the angular velocity must satisfy

\[ gT_2' - \nabla T_2 \left( g' + \frac{3}{r} g' \right) = 0. \]

If the axial velocity is steady \( T_1' = 0 \) and \( F(t) \) must be constant, and we then obtain

\[ f = c_1 r^2 + c_2 \log r + c_3, \]

which is finite on the axis if \( c_2 = 0 \). This furnishes the usual solution for Poiseuille flow

\[ u = u_0 (a^2 - r^2)/a^2. \]

If the axial velocity is not steady we must have \( T_1' / T_1 \) constant \( = -\gamma \lambda^2 \) say, and \( F(t) / T_1 \) constant. Hence (if we assume the latter constant to be zero for simplicity)

\[ T_1 = e^{-\gamma \lambda^2 t} \quad \text{and} \quad f(r) = k_1 J_0(\lambda r), \]

since \( f(r) \) must be finite on the axis; and \( u = 0 \) when \( r = a \) if \( J_0(\lambda a) = 0 \).

It appears therefore that there are possible motions of the form

\[ u = k_1 e^{-\gamma \lambda^2 t} J_0(\lambda r), \quad \text{where} \quad J_0(\lambda a) = 0. \]

If the angular motion is steady we obtain in the same way

\[ g = c_1 + c_3/r^2. \]

If it is not steady, and if we assume \( T_2 = T_1 \), the equation to determine \( g \) is
\[ rg'' + 3g' + \lambda^2 rg = 0, \]

and the solution

\[ g = k_2 r^{-2} \int r \cdot J_0(\lambda r) \, dr \]

is finite when \( r = 0 \). We can make \( u = w = 0 \) when \( r = a \) by taking

\[ g(r) = k_2 r^{-2} \int_r^a r \cdot J_0(\lambda r) \, dr \]

The additional term in \( r^{-2} \) due to the upper limit of integration represents an additional irrotational motion which leaves the vorticity unaltered, and does not affect \( \chi \) because it makes \( 2g + rg' = 0 \).

We have thus obtained a solution

\[ u = k_1 e^{-\nu \lambda^2 t} J_0(\lambda r), \]

\[ v = k_2 e^{-\nu \lambda^2 t} z r^{-2} \int_r^a r \cdot J_0(\lambda r) \, dr, \]

\[ w = -k_2 e^{-\nu \lambda^2 t} y r^{-2} \int_r^a r \cdot J_0(\lambda r) \, dr, \]

where \( J_0(\lambda a) = 0 \), which may be superimposed on the usual steady Poiseuille flow. The expressions for \( v \) and \( w \) are still valid solutions if in these definite integrals we take \( J_0(\lambda r) \) to be the general solution of Bessel's equation, for \( v \) and \( w \) will still vanish on the axis although \( g(r) \) is no longer finite there.

It may be verified without difficulty that the motions \((u, 0,0)\) and \((0, v, w)\) are not only ss. but also s. on each other.

7. **Ss. motions of type** \( \nabla \times U = \lambda U \).

Because of the importance of this type of motion it is a matter of considerable interest to inquire into the character of the scalar \( \lambda \).
It cannot be a function of \( t \) only, for in that case we should have form (6.4)

\[
\frac{\partial}{\partial t}(\lambda U) - \nu \nabla^2(\lambda U) = 0,
\]

i.e.

\[
\lambda \left( \frac{\partial}{\partial t} U - \nu \cdot \nabla^2 U \right) = -U \frac{d\lambda}{dt},
\]

while from (6.1)

\[
\left( \frac{\partial}{\partial t} - \nu \nabla^2 \right) U = \nabla \chi \quad \text{since} \quad \nabla \chi = 0.
\]

Combining these we should have

\[
U \frac{d\lambda}{dt} = \lambda \cdot \nabla \chi,
\]

which would mean that \( U \) is irrotational, a contradiction since \( \lambda \neq 0 \). Hence \( \lambda \) cannot be a function of \( t \) only.

If \( \nabla \chi = 0 \), so that Bernoulli’s equation has its usual form, \( \lambda \) cannot be a function of \( x, y, z \) other than a constant, i.e. \( \nabla \lambda = 0 \).

The equation of motion is now

\[
\left( \frac{\partial}{\partial t} - \nu \nabla^2 \right) U = 0.
\]

If \( \lambda \) depends on \( x, y, z \)

\[
\nabla \times (\nabla \times U) = \nabla \times (\lambda U) = \lambda (\nabla \times U) + \nabla \lambda \times U
\]

\[= \lambda^2 U + \nabla \lambda \times U,
\]

i.e.

\[
\nabla^2 U = -\lambda^2 U + U \times \nabla \lambda,
\]

and the equation of motion takes the form

\[
\left( \frac{\partial}{\partial t} + \nu \lambda^2 \right) U = \nu (U \times \nabla \lambda),
\]

and therefore

\[
U \cdot \frac{\partial U}{\partial t} + \nu \lambda^2 U^3 = U \cdot (U \times \nabla \lambda) = 0,
\]

so that

\[
U^3 = U_0^3 e^{-2 \nu \lambda^2 t},
\]

where \( U_0 \) depends only on \( x, y, z \), and \( U = U_0 e^{-\nu \lambda^2 t} \),
\[(\nabla^2 + \lambda^2) U_0 = 0. \] (7.1)

Hence \(U \times \nabla \lambda = 0,\)

and therefore either \(\nabla \lambda = 0,\) in which case the conclusion is established, or else \(U = \alpha \cdot \nabla \lambda,\)

where \(\alpha\) is a scalar. But from the relation \(\nabla \times U = \lambda U\)

it now follows that \(\nabla \times (\alpha \cdot \nabla \lambda) = \lambda \alpha \cdot \nabla \lambda\)

or \(\nabla \alpha \times \nabla \lambda = \lambda \alpha \cdot \nabla \lambda\)

which states that the vector \(\nabla \lambda\) is perpendicular to itself, a contradiction unless \(\nabla \lambda = 0.\)

The converse is also true. If \(\nabla \lambda = 0\) and \(\nabla \times U = \lambda U\) it follows that \(\nabla \chi' = 0.\)

If \(\nabla \lambda \neq 0\) all that can be said of \(\chi'\) is that it is a harmonic function.

The equations (7.1) indicate clearly the character of the motion when \(\lambda\) is a constant. General expressions for \(u, v, w\) have been given by Ram Ballabh (Superposable Fluid Motions: Proc. Benares Math. Soc. II, New Series, p. 76) in the form

\[
\begin{align*}
u &= e^{-\nu \lambda^2 t} (\varphi_x \cos \lambda z + \varphi_y \cos \lambda y + \chi_x \cos \lambda z + \chi_y \sin \lambda z), \\
v &= e^{-\nu \lambda^2 t} (\chi_y \cos \lambda z - \chi_x \sin \lambda z + \theta_y \cos \lambda x + \theta_x \sin \lambda x), \\
w &= e^{-\nu \lambda^2 t} (\theta_x \cos \lambda x - \theta_y \sin \lambda x + \varphi_x \cos \lambda y + \varphi_y \sin \lambda y).
\end{align*}
\] (7.2)

\[
\begin{align*}
\theta_x &= \theta_t = \theta_{xy} + \theta_{xz} = 0, \\
\varphi_y &= \varphi_t = \varphi_{xz} + \varphi_{xx} = 0, \\
\chi_x &= \chi_t = \chi_{xy} + \chi_{yy} = 0.
\end{align*}
\]

The motion represented by (7.2) may be regarded as the resultant of three plane motions of similar character. If we make \(\theta = \varphi = 0\)

\[
\begin{align*}
u &= e^{-\nu \lambda^2 t} (\chi_x \cos \lambda z + \chi_y \sin \lambda z), \\
v &= e^{-\nu \lambda^2 t} (\chi_y \cos \lambda z - \chi_x \sin \lambda z), \\
w &= 0, q^2 = e^{-2\nu \lambda^2 t} (\chi_x^2 + \chi_y^2)
\end{align*}
\] (7.3)
The magnitude of $q$ is independent of $z$, but its direction varies periodically with $z$ except when $\lambda = 0$, when the motion is irrotational.

The paths, which are stationary, are given by

$$
\frac{dx}{\chi_x \cos \lambda z + \chi_y \sin \lambda z} = \frac{dy}{\chi_y \cos \lambda z - \chi_x \sin \lambda z}
$$

which is immediately integrable if we introduce $\chi_h$, the harmonic conjugate of $\chi$, in the form

$$
\chi \sin \lambda z + \chi_h \cos \lambda z = c.
$$

The three component motions of this type are of course both s. and ss., since they are of the type $\nabla \times U = \lambda U$ with the same $\lambda$. Two motions of this type corresponding to the same $\chi$ but to different values of $\lambda$ satisfy the conditions of 3(g)(i), the planes $z = \text{constant}$ being the family of surfaces referred to in that paragraph. Two such motions will have the same stream lines in each of the planes

$$
z = 2n \pi / (\lambda_1 - \lambda_2)
$$

but not elsewhere. If two such motions have the same stream lines everywhere they differ only by a constant factor.

If $\psi$ is the stream function

$$
\psi = e^{-\nu \lambda^2 t} (\chi \sin \lambda z + \chi_h \cos \lambda z),
$$

$$
\nabla^2 \psi = 0, \quad \nabla^2 \psi = -\lambda^2 \psi.
$$

From the value of $q$ already given it follows that on a fixed boundary we must have simultaneously

$$
\chi_x = 0 \text{ and } \chi_y = 0.
$$

These conditions cannot be satisfied along any simple closed curve in the $(x, y)$ plane inside or outside which motion is taking place, for they imply

$$
\frac{\partial \chi}{\partial n} = 0
$$

at every point of the boundary, where $\partial n$ is an element of the normal; and since $\chi$ is a harmonic function it must be constant.
everywhere inside (or outside) the boundary, \( i.e. U = 0 \). Hence no such boundary can exist. If the conditions are satisfied at all it can only be at isolated points in the \((x, y)\) plane, \( i.e. \) on a set of isolated straight lines parallel to \( OZ \).

A similar statement is true for each of the three components of which the general motion is the resultant. If it is assumed that in order that the whole motion may vanish each of the components must do so separately, it follows that the equation \( q = 0 \) can be true, if at all, only at a set of points isolated in space, namely those points if any which are common to the three sets of straight lines parallel to the coordinate axes.

It may be thought at first sight that we can fulfil the requirements of a fixed boundary say on the plane \( z = 0 \) by means of

\[
\begin{align*}
  u &= e^{-\nu_2 t} \left( -\nu_2 x + \nu_2 \cos \lambda z + \nu_2 \sin \lambda z \right), \\
  v &= e^{-\nu_2 t} \left( -\nu_2 y + \nu_2 \cos \lambda z - \nu_2 \sin \lambda z \right), \\
  w &= 0.
\end{align*}
\]

This would be an error. The motions represented by

\[
\begin{align*}
  u_1 &= -\nu_2 x, & u_2 &= \nu_2 \cos \lambda z + \nu_2 \sin \lambda z, \\
  v_1 &= -\nu_2 y, & v_2 &= \nu_2 \cos \lambda z - \nu_2 \sin \lambda z, \\
  w_1 &= 0 & w_2 &= 0,
\end{align*}
\]

are not superposable. If motions \( s \), \( U_2 \) exist they must be rotational.

8. **Motions of type** \( \nabla \times U_1 = \lambda_2 U_2 \mid \nabla \times U_2 = \lambda_1 U_1 \) \( . \) (8.1)

These are always \( s \), with \( \nabla \chi = 0 \). The numbers \( \lambda_1, \lambda_2 \) are any scalars. On elimination we find

\[
\begin{align*}
  \nabla^2 U_1 + \frac{1}{\lambda_2} \nabla \nabla \lambda_2 \times (\nabla \times U_1) + \lambda_1 \lambda_2 U_1 &= 0, \\
  \nabla^2 U_2 + \frac{1}{\lambda_1} \nabla \nabla \lambda_1 \times (\nabla \times U_2) + \lambda_1 \lambda_2 U_2 &= 0.
\end{align*}
\]

These equations are identical if either

(i) \( \lambda_1 \) and \( \lambda_2 \) are independent of \( x, y, z \);

or (ii) \( \lambda_1 = \lambda_2 \).
In case (i) a general solution is easy. We have then to satisfy
\[(\nabla^2 + \lambda_1 \lambda_2) U = 0 \quad (8.3)\]
and
\[\nabla U = 0, \quad (8.4)\]
together with the equation of motion.

Let \(u_1, v_1\) be any two independent solutions of (8.3), and let
\[w_1 = \int_\alpha^z \left( \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) dz \quad (8.5)\]
where \(\alpha\) is independent of \(x, y, z\): so that (8.4) is satisfied identically by the choice of \(w_1\). Then
\[\nabla^2 w_1 = -\nabla^2 \int_\alpha^z dz \left( \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right)\]
\[= - \int_\alpha^z dz \left( \frac{\partial}{\partial x} \nabla^2 u_1 + \frac{\partial}{\partial y} \nabla^2 v_1 \right)\]
\[= \lambda_1 \lambda_2 \int_\alpha^z dz \left( \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) \quad \text{by (8.3)}\]
\[= - \lambda_1 \lambda_2 w_1 \quad \text{by (8.5)}.\]

Also from (8.1)
\[u_2 = \frac{1}{\lambda_2} \left( \frac{\partial w_1}{\partial x} - \frac{\partial v_1}{\partial z} \right),\]
\[v_2 = \frac{1}{\lambda_2} \left( \frac{\partial u_1}{\partial z} - \frac{\partial w_1}{\partial x} \right),\]
\[w_2 = \frac{1}{\lambda_2} \left( \frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y} \right),\]
i. e.
\[U_2 = \frac{1}{\lambda_2} (\nabla \times U_1).\]

which evidently makes \(\nabla U_2 = 0\) because \(\lambda_2\) does not depend on \(x, y, z\); and
\[ \nabla^2 U_2 = \frac{1}{\lambda^2} \nabla^2 (\nabla \times U_1) \]
\[ = \frac{1}{\lambda^2} \nabla \times \nabla^2 U_1 \]
\[ = -\lambda_1 (\nabla \times U_1) \quad \text{by (8.3)}, \]
\[ = -\lambda_1 \lambda_2 U_2 \quad \text{by (8.1)}, \]
so that the solution is complete provided that \( U_1 \) and \( U_2 \) satisfy the equation of motion.

If one of the two motions is ss. so is the other. For the respective ss. conditions are reduced on using (8.1) to
\[ \lambda_2 (U_1 \times U_2) = \nabla \chi_1 \quad \text{and} \quad \lambda_1 (U_2 \times U_1) = \nabla \chi_2, \]

When both are ss. the equations of motion may be written
\[ \frac{\partial}{\partial t} U_1 - \nu \nabla^2 U_1 = \nabla H_1, \]
\[ \frac{\partial}{\partial t} U_2 - \nu \nabla^2 U_2 = \nabla H_2, \]
where \( H_1 \) and \( H_2 \) denote harmonic functions. By (8.3) these become
\[ \frac{\partial}{\partial t} U_1 + \nu \lambda_1 \lambda_2 U_1 = \nabla H_1, \]
\[ \frac{\partial}{\partial t} U_2 + \nu \lambda_1 \lambda_2 U_2 = \nabla H_2, \]
hence \( U_1 = V_1 e^{-\nu \lambda_1 \lambda_2 t} \) an irrotational velocity, where \( V_1 \) does not depend on \( t \); and similarly for \( U_2 \).

Particular instances of case (ii) and of the general case when \( \lambda_1 \neq \lambda_2 \) can be constructed without difficulty.

In case (ii) the equation to be solved is
\[ (\nabla^2 + \lambda^2) U + \frac{\nabla \lambda}{\lambda} \times (\nabla \times U) = 0. \]

The operator \( \nabla. \) furnishes
\[ \nabla \cdot [\nabla (\log \lambda) \times (\nabla \times U)] = 0. \]

i.e.
\[ \nabla \lambda \cdot \nabla^2 U = 0, \]
and using the operator \( \nabla \lambda \) combined with this result \( U \cdot \nabla \lambda = 0 \).

For simplicity let \( \lambda_x = \lambda_y = 0 \), so that \( w = 0 \). We then find that \( u \) and \( v \) are both solutions of the equation
\[ \lambda \left( \nabla^2 + \lambda^3 \right) u - \lambda_x u_x = 0. \]

Introducing the stream function \( \psi \) in order to satisfy the continuity condition \( \nabla U = 0 \) we find \( u = -\psi_y, \ v = \psi_x \), and \( \psi \) is any solution of
\[ \lambda \left( \nabla^2 + \lambda^3 \right) \psi - \lambda_x \psi_x = 0, \]
of which particular solutions of the form \( \psi = AZ \) are easily obtained, where \( A \) is a function of \( x \) and \( y \) only, and \( Z \) of \( z \) only. This leads to
\[
\begin{align*}
Z' - \lambda^{-1} \lambda_x Z' + (\lambda^3 + c) Z = 0 \\
(\nabla_1^2 - c) A = 0.
\end{align*}
\]

(8.6)

We may take for example \( c = 0 \) and
\[
U_1 = Z(A_x i + A_y j + 0 k), \\
U_2 = \lambda^{-1} Z' (-A_y i + A_x j + 0 k),
\]
where \( i, j, k \) are as usual unit vectors along \( OX, OY, OZ \), and \( \nabla_1^2 A = 0 \).

Then
\[ \nabla \times U_1 = \lambda U_2 \]
and
\[ \nabla \times U_2 = -Z^{-1} \frac{d}{dz} (\lambda^{-1} Z') \cdot U_1 \]

Hence if we write \( \lambda_1 = \lambda \) and \( \lambda_2 = -Z^{-1} \frac{d}{dz} (\lambda^{-1} Z') \) it follows that
\[ \nabla \times U_1 = \lambda_1 U_2 \text{ and } \nabla \times U_2 = \lambda_2 U_1, \]
But the equation satisfied by \( Z \) shows of course that \( \lambda_1 = \lambda_2 \).

The motions \( U_1 \) and \( U_2 \) are not in general \( \lambda \).
9. When
\[ \nabla \times U_1 = \lambda_1 U_1 + \mu_1 U_2 \]
and
\[ \nabla \times U_2 = \lambda_2 U_2 + \mu_2 U_1 \]
(9.1)
a number of results are easily obtained similar to those of proceeding paragraphs. The s. condition is
\[ (\lambda_1 - \lambda_2) \,(U_1 \times U_2) = \nabla \lambda. \]
For brevity we shall regard \( \lambda_1, \lambda_2, \mu_1, \mu_2 \) as constants.
Using the operators \( U_1 \times \) and \( U_2 \times \) on the defining equations
\[ U_1 \times (\nabla \times U_1) = \mu_1 (U_1 \times U_2) \]
and \[ U_2 \times (\nabla \times U_2) = - \mu_2 (U_1 \times U_2); \]
hence if \( \lambda_1 \neq \lambda_2 \) and if \( U_2 \) s. \( U_1 \) it follows that both \( U_1 \) and \( U_2 \) are ss., and therefore that \( U_1 \pm U_2 \) are ss. Solutions of this kind can be superposed to any extent.

If \( \lambda_1 = \lambda_2 \) it is still true that
\[ \mu_2 U_1 \times (\nabla \times U_1) + \mu_1 U_2 \times (\nabla \times U_2) = 0. \]
(9.2)

There are three possibilities according as \( \mu_1 \) and \( \mu_2 \) are both zero, one is zero and the other not zero, or neither is zero.

If both are zero it is clear from (9.1) that \( U_1 \) and \( U_2 \) are both ss. of the type
\[ \nabla \times U = \lambda \, U \]
with the same value of \( \lambda \).

If \( \mu_1 = 0, \mu_2 \neq 0 \), \( U_1 \) is ss. of this type. The equation (9.2) is an identity and gives no information about \( U_2 \), which may or may not be ss., but if ss. it is not of type
\[ \nabla \times U = \lambda \, U. \]

If neither \( \mu_1 \) nor \( \mu_2 \) is zero it follows from (9.2) that either both \( U_1 \) and \( U_2 \) are ss. or neither is ss. If they are ss. they are not of type
\[ \nabla \times U = \lambda \, U. \]

The cases of interest are now those in which at least one of the motions is not ss.

If \( U_2 \) is not ss. we must have \( \lambda_1 = \lambda_2 = \lambda, \mu_2 \neq 0 \), and there are two cases according as \( \mu_1 \) is or is not zero.
Superposable Fluid Motions

When \( \mu_1 = 0 \), \( U_1 \) is ss. The defining equations are

\[
\nabla \times U_1 = \lambda \ U_1, \\
\nabla \times U_2 = \lambda \ U_2 + \mu \ U_1.
\]

Using the operator \( \nabla \times \) on the second equation we find

\[
(\nabla^2 + \lambda^2) \ U_2 = -2 \lambda \mu \ U_1,
\]
which determines \( U_2 \) when \( U_1 \) is given.

In particular when \( \lambda = 0 \) we must have

\[
\nabla^2 U_2 = 0.
\]

This agrees with 4.B.

In the general case when neither \( \mu_1 \) nor \( \mu_2 \) is zero we find that (on using the operator \( \nabla \times \)) both \( U_1 \) and \( U_2 \) are solutions of

\[
\nabla^2 U + 2 \lambda (\nabla \times U) + (\mu_1 \mu_2 - \lambda^2) U = 0,
\]
and on using \( \nabla \times \) again

\[
[(\nabla^2 + m)^2 + 4 \lambda^2 \nabla^2] U = 0,
\]

where \( m = \mu_1 \mu_2 - \lambda^2 \).

Any solution of (9.3) or (9.4), together with the \( U_3 \) determined uniquely by the defining equations, will serve as an example provided that both \( U_1 \) and \( U_2 \) render the equations of motion integrable.

10. There are not many exact solutions known in the theory of viscous fluid motion. It may be of interest to point out that most of them are in fact ss. solutions.

(1) When the motion of every particle of the fluid is parallel to the same fixed direction, say that of \( OZ \), the motion is always ss.

Since \( u = v = 0 \) it follows that

\[
\xi = w_y, \quad \eta = w_x, \quad \zeta = 0,
\]

\[
\chi_x = w w_x, \quad \chi_y = w w_y, \quad \chi_z = 0,
\]
and since \( w_z = 0 \) because of continuity.
\[ \chi = \frac{1}{2} w^2 = \frac{1}{2} U^2, \]

i.e. the motion is ss.

From (2.6) it follows that the quadratic terms in the equation of motion disappear identically, so that

\[ 0 = - \frac{\partial}{\partial x} \left( \frac{P}{\rho} + \Omega \right), \]
\[ 0 = - \frac{\partial}{\partial y} \left( \frac{P}{\rho} + \Omega \right), \]
\[ \frac{\partial}{\partial z} w - \nu \nabla^2 w = - \frac{\partial}{\partial z} \left( \frac{P}{\rho} + \Omega \right). \]

The operator \( \frac{\partial}{dz} \) gives \( \frac{\partial^3}{\partial z^3} \left( \frac{P}{\rho} + \Omega \right) = 0 \), (which is a particular case of the statement in (6.2) that \( \chi - \chi^1 \) is harmonic), i.e.

\[ \frac{P}{\rho} + \Omega = a + bz, \]

where \( a \) and \( b \) may be functions of the time, and

\[ \left( \frac{\partial}{\partial t} - \nu \nabla^2 \right) w = -b \]

This result includes the one dimensional flow of a viscous fluid under pressure between two parallel plates; flow along a straight tube of elliptic (or any other) cross section provided that the flow is parallel to the axis: and the flow due to the oscillation of a coordinate plane in a fixed direction parallel to itself, with a periodic velocity. All these are ss. whether steady or not.

(2) Viscous flow about a solid of revolution rotating about its axis with constant angular velocity \( \omega \).

The only exact solutions of which I am aware are those relating to circular cylinders, and that of the rotating disc (Goldstein, Modern Developments in Fluid Dynamics, I, p. 110).

The solution must be of the form
\[ u = -yg + xg, \]
\[ v = xf + yg, \]
\[ w = h, \]

where \( f, g, h \) are functions of \( z \) and \( r = (x^2 + y^2 + z^2)^{1/2} \). Since however \( g = h = 0 \) on the boundary of the rotating solid it is natural to inquire first whether or not there may exist purely rotatory ss. solutions of the form
\[ u = -yg, \]
\[ v = xf, \]
\[ w = 0. \]

The equation of continuity is identically satisfied. Proceeding as usual we obtain
\[ \xi = -x \left( \frac{z}{r} f_r + f_z \right), \]
\[ \eta = -y \left( \frac{z}{r} f_r + f_z \right), \]
\[ \zeta = 2f + \frac{1}{r} (r^3 - z^3) f_r, \]

where the suffixes denote differentiation with respect to explicit variables.

Also
\[ \chi_x = xf\zeta, \]
\[ \chi_y = yg\zeta, \]
\[ \chi_z = (r^3 - z^3) f \left( \frac{z}{r} f_r + f_z \right). \]

Now construct
\[ \psi = \frac{1}{2} U^3 - \chi = \frac{1}{2} (r^3 - z^3) f^2 - \chi. \]
\[ \psi_x = -xf\psi, \]
\[ \psi_y = -yg\psi, \]
\[ \psi_z = 0. \]
(This suffix covers both explicit and implicit variables z).

These equations are consistent only if \( f \) does not depend on \( z \) either explicitly or implicitly. It must therefore be either a constant (in which case the solid and fluid rotate together as a solid) or else a function of \( r_1 = (x^2 + y^2)^{1/2} \) say, and

\[
\psi = - \int r_1 f^2 dr_1, \quad \chi = \frac{1}{2} r_1 f^2 + \int r_1 f^2 dr_1.
\]

Given this, the motion is ss. whatever the form of the function \( f(r_1) \). But it is evident that since on the rotating boundary \( f = \omega \), then on that boundary \( r_1 \) is constant; no such ss. motion can therefore exist unless the boundaries are coaxial circular cylinders rotating about the axis of figure. There is in particular no such ss. motion in the case of a sphere or a rotating disc. In these cases the only possible ss. motion of this purely rotatory kind is when \( f = \omega \) everywhere.

It is easily shown that in the case of circular cylindrical boundaries we obtain precisely the usual solutions.

It may also be shown from the equations of motion that these also require that \( f \) be a function of \( r_1 \) only, so that except in the case of the rotation of solid and fluid as a solid, and that of circular cylindrical boundaries, the fluid in the neighbourhood of a steadily rotating solid of revolution cannot have a purely rotatory motion about the axis, whether ss. or not.

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