Unitary up to a factor Representations of the inhomogeneous Galilei group and the non-relativistic Schrödinger equation.

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Özet.— Dalga mekaniğinde bir $\Phi$ hali muhtelif $g$, $g'$ koordinat sistemlerinde $\Phi_g, \Phi_{g'}$ gibi birbirinden farklı dalga fonksiyonlarıyla gösterilir. İki $\Phi, \Psi$ hali arasındaki atlama ihtimalinin (transition probability) değişimliği sebebiyle, $\Phi_g, \Psi$ nun zaten itibari olan faz o şekilde seçilebilir ki, bütün $\Phi_g, \Phi_{g'}$ ler için $\Phi_g = D(N)\Phi_{g'}$ cari olur; burada $D(N)$ lineer ve uniter bir operatör ve $N$, $g$ sistemini $g'$ sistemine götüren transformasyondur ($g' = N g$). Rölativistik olmayan dalga mekaniğinde genel Galile transformasyonlarına karşı değişimlik düşünüldüğü için, $D(N)$ operatörleri inhomogen Galile grubunun bir çarpan farklıyla bir röprezantasyonunu meydana getirirler; yani

$$D(N_1) D(N_2) = w(N_1, N_2) D(N_1 N_2)$$

olur ki burada $w(N_1, N_2)$ nun mutlak değeri 1 dir. Wigner, Lorentz grubu için bu operatörleri $w = \mp 1$ olacak şekilde normalmak mümkün olduğunu göstermişti. Bu yazarda Wigner’in metodu bizi ilgilendiren Galile grubuna tabtıkt edilmektedir. Normlama sonucunda bir sabit farklıyla belirli özel bir röprezantasyon elde edildiği ve bu röprezantasyonun da esas itibariyle rölativistik olmayan Schrödinger denkleminin düzlem-dalgı (plane-wave) çözümlerinin meydana getirdiği röprezantasyondan başka bir şey olmadığı görülcektir.

**

Introduction. The inhomogeneous proper Galilei group contains, in addition to proper Galilei transformations (given by $x' = x + vt$), displacements of the origin both in space (given by $x' = x + a$) and in time ($t' = t + b$). Consider all the frames of reference that can be obtained from each other by transformations of this group (ie By an inhomogeneous Galilei transformation). In non-relativistic quantum mechanics, any two such frames should be physically equivalent. Since the transition probability between two states $\Phi$ and $\Psi$, defined
as the square of the modulus of the unitary scalar product 
\((\Phi, \Psi)\) of the two normalized wave functions \(\Phi, \Psi\) has an
invariant physical meaning, it must have the same value in both
frames. Thus if the states are described by \(\Phi_g, \Psi_g\) in the \(g\)
frame and by \(\Phi_{g'}, \Psi_{g'}\) in the \(g'\) frame, one must have

\[
|\langle \Phi_g, \Psi_g \rangle|^2 = |\langle \Phi_{g'}, \Psi_{g'} \rangle|^2
\]

By an appropriate choice of the physically meaningless
constants in \(\Phi_g\), one can derive from (1), [1], the existence
of a linear unitary operator \(D(N)\) such that

\[
\Phi_{g'} = D(N) \Phi_g
\]

for all functions \(\Phi_g, \Phi_{g'}\), where \(N\) is the transformation that
carries \(g\) into \(g' = Ng\). The operator \(D(N)\) is determined by
the physical content of the theory only up to a constant of
modulus unity which can depend on \(g\) and \(g'\). Consequently
the \(D(N)\) form a representation up to a factor of the inhomogeneous
Galilei group:

\[
D(N_1) D(N_2) = \omega(N_1, N_2) D(N_1N_2)
\]

where \(\omega\) is a number whose phase can depend on \(N_1, N_2\) but
whose modulus is equal to unity. This whole argument is
taken bodily from the discussion given by Wigner for relativistic
quantum mechanics in his paper on «Unitary representations of
the inhomogeneous Lorentz group» [2], where he
shows that the operators which transform relativistic wave
functions in different Lorentz frames into each other form a
representation up to a factor of the inhomogeneous Lorentz
group.

In the case of the Lorentz group, by making use of the
mathematical properties of the group, Wigner could further
show that it is possible to give a definite phase to each ope-
\(I\) which leaves only the sign undetermined and thus
obtain for these normalized operators \(U(L)\),

\[
U(L_1) U(L_2) = \mp U(L_1L_2).
\]

In this note we want to apply his mathematical arguments to
the case of the Galilei group and see to what extent the ge-
neral representations up to a factor can be simplified by a
proper normalization. It will be seen that such a normaliza-
tion is still fruitful and leads essentially to the special rep-
presentation up to a factor formed by the plane-wave solutions of the non-relativistic Schrödinger equation. Prof. Bargmann has independently obtained the same result by a more general method [3].

After discussing briefly the Galilei group we shall carry out the normalization step by step, following closely Wigner’s method. In fact some of the arguments that he developed for the Lorentz group are equally valid for the Galilei group; in those cases we shall simply refer to his paper (Ref. 2).

**Description of the inhomogeneous Galilei group.** We shall denote the general element of the proper inhomogeneous Galilei group by \( N = (a, b, v, R) \) where \( a \) represents the space translation \( x' = x + a \), \( b \) represents the time displacement \( t' = t + b \), \( v \) represents the uniform acceleration \( x' = x + vt \) and \( R \) represents the rotation \( x' = Rx \). The order of the transformations is from right to left. By direct substitutions one easily obtains the following relations between the elementary transformations:

\[
\begin{align*}
(4) & \quad (a_1)(a_2) = (a_2)(a_1) = (a_1 + a_2) \\
(5) & \quad (b_1)(b_2) = (b_2)(b_1) = (b_1 + b_2) \\
(6) & \quad (v_1)(v_2) = (v_2)(v_1) = (v_1 + v_2) \\
(7) & \quad (R_1)(R_2) = (R_2R_1) \\
(8) & \quad (a)(b) = (b)(a) \\
(9) & \quad (a)(v) = (v)(a) \\
(10) & \quad (R)(a) = (a')(R) \quad \text{where } a' = Ra \\
(11) & \quad (R)(b) = (b)(R) \\
(12) & \quad (R)(v) = (v')(R) \quad \text{where } v' = Rv \\
(13) & \quad (v)(b) = (b)(a)(v) \quad \text{where } a = bv.
\end{align*}
\]

By means of the relations (4-13), the product of two inhomogeneous Galilei transformations is easily calculated to be another such transformation:

\[
(a_1, b_1, v_1, R_1)(a_2, b_2, v_2, R_2) = (a_{12}, b_{12}, v_{12}, R_{12})
\]

where

\[
\begin{align*}
a_{12} & = a + b_2v_1 - R_1a_2 \\
b_{12} & = b_1 + b_2 \\
v_{12} & = v_1 + R_1v_2 \\
R_{12} & = R_1R_2.
\end{align*}
\]

(14)
Operators. The operator $D(N)$ for the general transformation can be decomposed into four elementary operators corresponding to the four elementary transformations:

$$D(N) = T(a) \theta(b) G(v) O(R).$$

Then using the commutation relations (4-13), we obtain from (2) the equivalent relations:

(16) \hspace{1cm} T(a_1) T(a_2) = \omega(a_1, a_2) T(a_1 + a_2)

(17) \hspace{1cm} \omega(b_1, b_2) \theta(b_1 + b_2)

(18) \hspace{1cm} G(v_1) G(v_2) = \omega(v_1, v_2) G(v_1 + v_2)

(19) \hspace{1cm} O(R_1) O(R_2) = \omega(R_1, R_2) O(R_1 R_2)

(20) \hspace{1cm} O(R) T(a) = \omega(R, a) T(Ra) O(R)

(21) \hspace{1cm} O(R) G(v) = \omega(R, v) G(Rv) O(R)

(22) \hspace{1cm} O(R) \theta(b) = \omega(R, b) \theta(b) O(R)

(23) \hspace{1cm} T(a) \theta(b) = \omega(a, b) \theta(b) T(a)

(24) \hspace{1cm} T(a) G(v) = \omega(a, v) G(v) T(a)

(25) \hspace{1cm} G(v) \theta(b) = \omega(v, b) \theta(b) T(bv) G(v).

Normalization. The purpose of the normalization is to eliminate the arbitrariness in the $\omega$'s of (16-25) as much as possible. To this end we point out the following theorems.

I. All $T(a)$ commute.

Proof: From (16) we have

$$T(a_2) = T(a_2) T(a_1) T(a_1)^{-1} = \omega(a_2, a_1) \omega(a_2 + a_1, -a_1) T(a_2)$$

or

$$\omega(a_1 + a_2, -a_1) = \omega(a_2, a_1)^{-1}.\tag{26}$$

Hence

$$T(a_1) T(a) T(a_1)^{-1} = \frac{\omega(a_1, a_2)}{\omega(a_2, a_1)} T(a_2) = c(a_1, a_2) T(a_2).\tag{27}$$

with

$$c(a_1, a_2) = c(a_2, a_1)^{-1}.\tag{28}$$

Transforming (27) with $T(a_3)$ we obtain

$$T(a_3) T(a_1) T(a_2) T(a_1)^{-1} T(a_3)^{-1} = c(a_1, a_2) T(a_3) T(a_2) T(a_3)^{-1}$$

or

$$\omega(a_3, a_1) T(a_3 + a_1) T(a_2) w(a_3, a_1)^{-1} T(a_3 + a_1)^{-1} = c(a_1, a_2) c(a_3, a_2) T(a_2)$$

or
It follows from (29) that (Ref. 2. page 171)

\begin{equation}
(30) \quad c(a_1, a_2) = \exp \left\{ 2\pi i \sum_{x=1}^{3} a_{1x} f_x(a_2) \right\}
\end{equation}

and using (28) we obtain

\begin{equation}
(31) \quad \sum_{x=1}^{3} \left[ a_{1x} f_x(a_2) + a_{2x} f_x(a_1) \right] = n(a_1, a_2)
\end{equation}

where \( n(a_1, a_2) \) is an integer. Setting for \( a_3 \) in (31) the three unit vectors \( e_\lambda \) (\( \lambda \) component of which is 1, the others being zero) in turn and letting \( f_x(e_\lambda) = - f_{x\lambda} \) yields

\begin{equation}
(32) \quad f_x(a_1) = n(a_1, e_\lambda) + \sum_{x=1}^{3} a_{1x} f_{x\lambda}
\end{equation}

and putting this back into (31) we find

\begin{equation}
(33) \quad f_{x\lambda} + f_{\lambda x} = 0 \quad \text{and} \quad n(a, e_\lambda) = 0.
\end{equation}

Now, by assuming for the components of \( a_1 \) and \( a_2 \) such values that are transcendental both with respect to each other and the \( f_{x\lambda} \) (which are fixed numbers) we see that (32) can not hold except if the coefficient of every term vanishes; i.e.

\begin{equation}
(34) \quad c(a_1, a_2) = \exp \left\{ 2\pi i \sum_{x, \lambda=1}^{3} a_{1x} a_{2\lambda} f_{x\lambda} \right\}
\end{equation}

We now transform the equation (27) by the operator \( O(R) \)

\begin{equation}
O(R) \ T(a_1) \ T(a_2) \ T(a_1)^{-1} \ O(R)^{-1} = c(a_1, a_2) \ w(R, a_2) \ T(Ra_2),
\end{equation}

on the other hand,

\begin{align*}
O(R) \ T(a_1) \ O(R)^{-1} O(R) \ T(a_2) \ O(R)^{-1} O(R) \ T(a_1)^{-1} O(R)^{-1} \\
= w(R, a_1) \ T(Ra_1) \ w(R, a_2) \ T(R, a_2) \ w(R, a_1)^{-1} \ T(Ra_1)^{-1} \\
= w(R, a_2) \ c(Ra_1, Ra_2) \ T(Ra_2)
\end{align*}

hence

\begin{equation}
(35) \quad c(a_1, a_2) = c(Ra_1, Ra_2)
\end{equation}
for any rotation $R$. Combined with (34) this gives,

$$\sum_{\kappa, \lambda=1}^{3} \left( f_{x \kappa} a_{1 \kappa} a_{2 \lambda} - \sum_{\mu, \nu=1}^{3} f_{\mu \nu} R_{\mu \kappa} R_{\nu \lambda} a_{1 \mu} a_{2 \nu} \right) = n'(a_1, a_2)$$

where $n(a_1, a_2)$ is an integer. Since this is true for every $a_1, a_2$, we again obtain

$$f_{x \kappa} = \sum_{\mu, \nu=1}^{3} f_{\mu \nu} R_{\mu \kappa} R_{\nu \lambda}$$

or

$$f = R' f R = R^{-1} f R$$

for any rotation. As the only matrix which is invariant under all rotations is the identity matrix, it follows from (33) that $f$ vanishes identically.

Hence

$$c(a_1, a_2) = 1$$

and

$$T(a_1) T(a_2) = T(a_2) T(a_1).$$

II. All $G(v)$ commute. Proof is identical with that for I.

III. All $\Theta(b)$ commute. Proof is very similar. Instead of (30), (31) we obtain in the same way,

$$c(b_1, b_2) = \exp \left\{ 2\pi i b_1 f(b_2) \right\}$$

where $f$ is a scalar and

$$b_1 f(b_2) + b_2 f(b_1) = n(b_1, b_2)$$

where $n(b_1, b_2)$ is an integer; this gives for $b_1 = b_2 = b$

$$f(b) = \frac{n(b_1 b)}{2b}$$

and putting it back into (38) we have

$$\frac{b_1}{b_2} n(b_2, b_2) + \frac{b_2}{b_1} n(b_1, b_1) = 2n(b_1, b_2)$$

By choosing a transcendental value for $\frac{b_1}{b_2}$, we see that this equation can only be satisfied if

$$n(b, b) = 0, \quad n(b_1, b_2) = 0.$$
Consequently
\[ f(b) = 0, \quad c\left(b_1, b_2\right) = 1 \]
and
\[ \Theta(b_1) \Theta(b_2) = \Theta(b_2) \Theta(b_1). \]

**IV. Normalization of \( \Theta(b) \).** Prof. Bargmann has shown very simply how every \( \Theta(b) \) can be multiplied with a definite phase factor \( e^{-i \eta(b)} \) so as to get \( w(b_1, b_2) = 1 \) for the new operators
\[ V(b) = e^{-i \eta(b)}(b). \]

We shall reproduce here his proof for the sake of completeness.

By multiplying with the phase factors we obtain from (17)
\[ V(b_1) V(b_2) = w'(b_1, b_2) V(b_1, b_2) \]
where
\[ w'(b_1, b_2) = w(b_1, b_2) \exp \left\{ - i \left[ \eta(b_1) + \eta(b_2) - \eta(b_1 + b_2) \right] \right\} \]
with
\[ w(b_1, b_2) = w(b_2, b_1); \]
or letting
\[ w(b_1, b_2) = \exp \{i \xi(b_1, b_2)\} \]
we have
\[ \xi'(b_1, b_2) = \xi(b_1, b_2) - \eta(b_1) - \eta(b_2) + \eta(b_1 + b_2). \]

The function \( \xi(b_1, b_2) \) satisfies two relations:
\[ \xi(b_1, b_2) = \xi(b_2, b_1); \]
which follows from (42); and
\[ \xi(b_1, b_2) + \xi(b_1 + b_2, b_3) = \xi(b_1, b_2 + b_3) + \xi(b_2, b_3) \]
which follows from the associativity of \( \Theta(b) \).

We now claim that for any given continuous \( \xi(b_1, b_2) \) that satisfies (43-44), one can find a function \( \eta(b) \) such that
\[ \xi'(b_1, b_2) = 0 \quad \text{or} \quad \xi(b_1, b_2) = \eta(b_1) + \eta(b_2) - \eta(b_1 + b_2). \]

To prove this assertion, we remark first that if \( \eta_0(b) \) satisfies (45), so does \( \eta(b) = \eta_0(b) + Cb \) where \( C \) is any constant. In particular for \( C = - \eta_0(1) \) we have
\[ \eta(1) = 0. \]

We shall suppose that (46) is satisfied. Integrating (45) with respect to \( b_2 \) and interchanging \( b \) and \( b_1 \) we obtain
\[
\int_0^b \xi(b_1, b_2) \, db_2 = b\eta(b_1) + \int_0^b \eta(b_2) \, db_2 - \int_0^b \eta(b_1 + b_2) \, db_2 \\
\int_0^{b_1} \xi(b_1, b_2) \, db_2 = b_1\eta(b) + \int_0^{b_1} \eta(b_2) \, db_2 - \int_0^{b_1} (b + b_2) \, db_2
\]

and after subtracting the first equation from the second and setting \(b_1 = 1\),

\[
(47) \quad \eta(b) = \int_0^1 \xi(b, b_2) \, db_2 - \int_0^b \xi(1, b_2) \, db_2.
\]

It remains to show that (47) indeed satisfies (45). We have

\[
\eta(b_1) + \eta(b_2) - \eta(b_1 + b_2) = \int_0^1 \left[ \xi(b_1, b) + \xi(b_2, b) - \xi(b_1 + b_2, b) \right] \, db \\
- \int_0^{b_1} \xi(1, b) \, db - \int_0^{b_2} \xi(1, b) \, db + \int_0^{b_1 + b_2} \xi(1, b) \, db
\]

or by using (44),

\[
= \xi(b_1, b_2) + F(b_1, b_2)
\]

where

\[
F(b_1, b_2) = \int_0^1 \xi(b_1, b) \, db - \int_0^{b_2} \xi(b_1, b) \, db + \int_0^{b_1 + b_2} \xi(1, b) \, db - \int_0^{b_2} \xi(1, b) \, db;
\]

but since

\[
F(b_1, 0) = 0
\]

and again by (44)

\[
\frac{\partial F(b_1, b_2)}{\partial b_2} = -\xi(b_1, b_2 + 1) + \xi(b_1, b_2) + \xi(1, b_1 + b_2) - \xi(1, b_2) = 0
\]

we must have

\[
F(b_1, b_2) \equiv 0
\]

which proves the theorem. Consequently we shall put in what follows

\[
(48) \quad \omega(b_1, b_2) = 1.
\]
V. Normalization of $T(a)$. Consider the unit vectors $e_i$, $e_2$, $e_3$ in the $x$, $y$, $z$ directions. The arguments of the previous section show that one can normalize $T(a_1e_1)$, $T(a_2e_2)$, $T(a_3e_3)$ in such a way that

\[(49) \quad T(a_i e_i) T(b_i e_i) = T[(a_i + b_i)e_i], \quad i = 1, 2, 3.\]

One can also fix the equality

\[(50) \quad T(a) = T(a_1e_1 + a_2e_2 + a_3e_3) = T(a_1e_1)T(a_2e_2)T(a_3e_3)\]

as this essentially defines the operation $T(a)$. Then we shall have using (49), (50) and (36),

\[
T(a)T(b) = T(a_1e_1)T(a_2e_2)T(a_3e_3)T(b_1e_1)T(b_2e_2)T(b_3e_3)
= T[(a_1 + b_1)e_1]T[(a_2 + b_2)e_2]T[(a_3 + b_3)e_3] = T(a + b);
\]

hence

\[(51) \quad w(a, b) = 1.\]

The equation $T(a)T(b) = T(a + b)$ remains valid if one replaces $T(a)$ by $e^{2\pi i a \cdot c}T(a)$, where $c$ is an arbitrary constant vector. Following Wigner we shall make use of this remaining freedom to eliminate $w(R, a)$. From (20) we have

\[(52) \quad O(R)T(a)O(R)^{-1} = w(R, a)T(Ra)\]

and transforming it with $O(S)$,

\[
O(S)O(R)T(a)O(R)^{-1}O(S)^{-1} = w(R,a)O(S)T(Ra)O(S)^{-1}
\]

which gives, using (19),

\[
Rw(S,R)O(SR)T(a)w(S,R)^{-1}O(SR)^{-1} = w(R,a)w(S,Ra)T(SRa)
\]

or

\[(53) \quad w(SR,a) = w(R,a)w(S,Ra).\]

On the other hand, applying $O(R)$ to $T(a_1)T(a_2) = T(a_1 + a_2)$ we obtain

\[
O(R)T(a_1)T(a_2) = w(R,a_1)T(Ra_1)O(R)T(a_2)
= w(R,a_1)w(R,a_2)T(Ra_1)T(Ra_2)O(R)
\]

or

\[(54) \quad w(R,a_1 + a_2) = w(R,a_1)w(R,a_2),\]

therefore

\[(55) \quad w(R,a) = \exp \{2\pi i a \cdot f(R)\}, \quad \text{where} \ f \ \text{is a vector}.\]
Inserting this back into (53) we obtain
\[ a \cdot \{ f(SR) - f(R) - R^{-1}f(S) \} = n \]
where \( n \) is integer; which shows, since \( n \) depends linearly on the arbitrary vector \( a \), that \( n = 0 \) and
\[ f(SR) = f(R) + R^{-1}f(S). \]
(56)

It is shown in Ref. 2 (page 175) that (56) is equivalent to
\[ f(R) = (1 - R^{-1})r_0 \]
where \( r_0 \) is an arbitrary constant vector. We thus obtain
\[ w(R, a) = \exp \left\{ 2\pi i a \cdot (1 - R^{-1})r_0 \right\} \]
and
\[ O(R)T(a) = \exp \left\{ 2\pi i a \cdot (1 - R^{-1})r_0 \right\} T(Ra)O(R) \]
or, after multiplying every \( T(a) \) by \( \exp \left\{ -2\pi i a \cdot r_0 \right\} \), for the new operators
\[ (57) \quad O(R)T(a) = T(Ra)O(R). \]

This completes the normalization of \( T(a) \).

VI. Normalization of \( G(v) \) is done in the same way as in \( V \) and results are
\[ (58) \quad w(v_1, v_2) = 1; \quad w(R, v) = 1. \]

VII. \( \Theta(b) \) commutes with \( O(R) \).

Proof: By applying the same method as the one used in eliminating \( w(R, a) \) we obtain the two relations:
\[ (59) \quad w(R, b_1 + b_2) = w(R, b_1) + w(R, b_2) \]
\[ (60) \quad w(R_2R_1, b) = w(R_1, b)w(R_2, b). \]

From (59) follows
\[ w(R, b) = \exp \left\{ 2\pi ibf(R) \right\} \]
where \( f \) is a scalar. Inserting this in (60) we obtain as usual,
\[ (61) \quad f(R_2R_1) = f(R_2) + f(R_1). \]

Consequently we have
\[ (62) \quad f(R^n) = nf(R) \quad \text{and} \quad f(E) = 0 \]
where \( n \) is a positive integer and \( E \) is identity matrix. It fol-
lows from (62) that if \( R_q \) is a rotation around any axis by an angle \( 2\pi \frac{p}{q} \) (\( p, q \) being positive integers) we have

\[
q f(R_q) = 0 \quad \text{or} \quad f(R_q) = 0.
\]

Therefore \( f(R) \) must vanish identically and we have

\[
w(R, b) = 1.
\]

**VIII. \( \Theta(b) \) commutes with \( T(a) \).**

**Proof:** From (23) we obtain as previously,

\[
w(a, b) w(a_2, b) = w(a_1 + a_2, b)
\]

\[
w(a, b_1) w(a, b_2) = w(a, b_1 + b_2).
\]

and the resulting equations

\[
w(a, b) = \exp \{2\pi i a \cdot f(b)\}
\]

where \( f \) is a vector; and

\[
f(b_1 + b_2) = f(b_1) + f(b_2).
\]

The general solutions of (66) is

\[
f(b) = bB
\]

where \( B \) is an arbitrary constant vector. We have now

\[
T(a) \Theta(b) = \exp \{2\pi i b \cdot A\} \Theta(b) T(a).
\]

However, transforming with \( O(R) \) we can see that \( B = 0 \); it gives in fact

\[
O(R) T(a) \Theta(b) O(R)^{-1} = \exp \{2\pi i b \cdot A\} O(R) \Theta(b) T(a) O(R)
\]

or using \( V. \) and VII.,

\[
T(Ra) \Theta(b) = \exp \{2\pi i b \cdot A\} \Theta(b) T(Ra);
\]

but we also have from (67) directly

\[
T(Ra) \Theta(b) = \exp \{2\pi i b \cdot Ra \cdot B\} \Theta(b) T(Ra).
\]

Comparing (68) and (69) we obtain, since \( b \) is arbitrary,

\[
a \cdot B = Ra \cdot B \quad \text{or} \quad a \cdot (B - R^{-1}B) = 0 \quad \text{for every} \ a;
\]

hence

\[
RB = B \quad \text{for every} \ R.
\]

Consequently

\[
B \equiv 0
\]
and
\( w(a, b) = 1 \).

\( \text{IX. Normalization of } O(R) \) It is shown in Ref. 2 (pages 176-178) that \( O(R) \) can be normalized so as to give
\( w(R_1, R_2) = +1 \).

Because of the form of the relations (19-22), this normalization does not interfere with other normalizations.

\( \text{X. Determination of } w(a, v) \).

From (24) we obtain, in the usual way,
\[
\begin{align*}
(w(a_1, v)w(a_2, v)) & = w(a_1 + a_2, v) \\
(w(a_1v_1)w(a, v_3)) & = w(a, v_1 + v_3)
\end{align*}
\]
from which follows
\[
w(a, v) = \exp \{2\pi i a \cdot f(v)\}
\]
where \( f \) is a vector satisfying
\( f(v_1 + v_2) = f(v_1) + f(v_2) \).
The general solution of (74) is given by
\[
f(v) = Av
\]
where \( A \) is a constant arbitrary matrix. Hence (24) becomes
\( T(a)G(v) = \exp \{2\pi i a \cdot Av\}G(v)T(a) \).

However, as in VIII., this expression is not symmetrical enough; in fact transforming with \( O(R) \) we obtain
\[
T(Ra)G(Rv) = \exp \{2\pi i a \cdot Av\}G(Rv)T(Ra)
\]
which combined with (75) gives
\[
a \cdot Av = Ra \cdot ARv = a \cdot R^{-1}ARv \quad \text{for every } a \text{ and } v;
\]
or
\[
A = R^{-1}AR \quad \text{for every } R.
\]
It follows that \( A \) is a multiple of the unit matrix and can be taken as an ordinary number; we have thus finally,
\( w(a, v) = \exp \{2\pi i Aa \cdot v\} \)
where \( A \) is an arbitrary constant.

\( \text{XI. Determination of } w(v, b) \).

From (25) we have,
\[
G(v_1)\Theta(b)G(v_4)^{-1} = w(v_1, b)\Theta(b)T(bv_1);
\]
transforming this with $G(v_2)$ we obtain,

$$G(v_2)G(v_1)\theta(b)G(v_1)^{-1}G(v_2)^{-1} = w(v_1, b)G(v_2)\theta(b)T(bv_1)G(v_2)^{-1}$$

or using the relations established so far,

(79) \[ w(v_1 + v_2, b) = w(v_1, b_1)w(v_2, b_2) \exp \left\{ -2\pi i A b v_1 \cdot v_2 \right\}. \]

On the other hand by means of

$$G(v)\theta(b_1)\theta(b_2) = w(v, b_1)\theta(b_1)T(b_1v)w(v, b_2)\theta(b_2)T(b_2v)G(v)$$

we find

(80) \[ w(v, b_1 + b_2) = w(v, b_1)w(v, b_2) \]

from which follows

$$w(v, b) = \exp \left\{ 2\pi i b f(v) \right\}$$

where $f$ is a scalar. Inserting this in (79) we find

(81) \[ f(v_1 + v_2) = f(v_1) + f(v_2) - A v_1 \cdot v_2 \]

The general solution of (81) is given by

$$f(v) = -\frac{A}{2} v^2 + v \cdot V_0$$

but again by operating with $O(R)$ one can show that $V_0 = 0$. Thus we have

(82) \[ w(v, b) = \exp \left\{ -2\pi i A \frac{b}{2} v^2 \right\}. \]

**Conclusion.**

Using (48), (51), (57), (58), (63), (70), (71), (77) and (82) we have for the normalised operators

$$D(N_1)D(N_2) = T(a_1)\theta(b_1)G(v_1)O(R_1)T(a_2)\theta(b_2)G(v_2)O(R_2)$$

$$= \exp \left\{ -2\pi i A (a_2 \cdot v_1 + \frac{b_2}{2} v_1^2) \right\} T(a_1 + b_2v_1 + R_1a_2)\theta(b_1 + b_2)$$

$$G(v_1 + Rv_2)O(R_1R_2)$$

or by (14)

(83) \[ D(N_1)D(N_2) = \exp \left\{ -2\pi i A (a_2 \cdot v_1 + \frac{1}{2} b_2v_1^2) \right\} D(N_1N_2) \]

where $A$ is an arbitrary constant. This shows that by proper normalization all the representations up to a factor of the Galilei group (of the form (2) where $w(N_1, N_2)$ is of modulus unity) can be brought to the same form. They will only differ from
each other by the value of the arbitrary constant $A$. Furthermore, (83) is essentially the representation formed by the plane-wave solutions of the non-relativistic Schrödinger equation. Consider in fact the plane-wave solution for a particle with mass $m$, momentum $p$ and spin zero,

\begin{equation}
\Psi(p) = \exp \left\{ \frac{2\pi i}{\hbar} \left( p \cdot x - \frac{p^2}{2m} t \right) \right\}
\end{equation}

One easily obtains for this solution

\begin{align}
T(a)\Psi(p) &= \exp \left\{ -\frac{2\pi i m}{\hbar} a \cdot v \right\} \Psi(p) \\
\Theta(b)\Psi(p) &= \exp \left\{ \frac{2\pi i m}{\hbar} b \right\} \Psi(p) \\
G(v)\Psi(p) &= \Psi(p - mv) \\
O(R)\Psi(p) &= \Psi(R^{-1}p)
\end{align}

and consequently

\begin{equation}
D(N_1)D(N_2)\Psi(p) = \exp \left\{ 2\pi i \frac{m}{\hbar} (a_2 \cdot v_1) \right\} D(N_1N_2)\Psi(p).
\end{equation}

This representation is identical with the one obtained from (83) by letting $A = -\frac{m}{\hbar}$ and taking the positive sign for zero spin.

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References


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