On The Chebyshev Approximation by $A + B^* \log (1 + CX)$

by

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ABSTRACT

Previous studies on the Chebyshev approximation are enlightened, and the best Chebyshev approximation proved to be $A + B^* \log (1 + CX)$ on $[0, \infty]$ and it is generalized with the help of new concepts.

INTRODUCTION

The most general approximation problem, first presented in 1970 by Barrodal [1], can be express shortly as the following:

On the assumption that $X$ is a topologic space and $C(X)$ a set of bounded and continuous functions (have real and complex values) on space $X$, $C(X)$ space can be set up by norm

$$
\|g\| = \sup\{ |g(x)| ; x \in X \}
$$

Let $P$ be a parameter space and $F$ approximation function in $C(X)$ corresponding an element $A$ of parameter space $P$ such as $F(A, \cdot) = F[A]$. There is an element, $F[A]$, for $f$ which is in $C(X)$ such that

$$
\rho(f, X) = \inf \{ \|f - F[A^*]\| ; A \in P \}
$$

with the condition of

$$
\rho(f, X) = \|f - F[A^*]\|
$$

then $A$ is called “best parameter” and the function $F[A^*]$ “best approximation” to $f$ on $X$. Searching $A^*$ is the essential of Chebyshev problem.

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Solution of Chebyshev approximation problem is carried out by means of varying $X$, $F$ and $P$. The conditions hold in for the solution of Chebyshev problem are important.

G. Meinardus and Schwedt [2] found out important theorems in 1964 which are used for the best approximation in Chebyshev problem. Then many scientists have studied on Chebyshev approximation problem under various conditions [3]. C.B. Dunham [4], [5] proved that the best approximation would be $A + B^* \log (1 + CX)$ on $[0, x]$.

In our study we set up new lemmas, theorems and definitions in order to enlighten the obscurities in previous studies and to prove the best Chebyshev approximation to be $A + B^* \log (1 + CX)$ on $[0, x]$. Furthermore, we have generalized it by means of new concepts.

EXTENSIVE SOLUTION OF CHEBYSHEV APPROXIMATION BY $A + B^* \log (1 + CX)$

Topologic concepts are invariant under an homomorphism. $[-1, +1]$ is homomorph to $[0, x]$ so we can use $[-1, +1]$ instead of $[0, x]$.

Let $C([-1, +1])$ be the space of defined and numerical functions on $[-1, +1]$ with norm

$$
\|g\| = \sup \{|g(x)|; -1 \leq x \leq +1\}
$$

and with the condition

$$
P = \{A: A = (a_1, a_2, a_3) \in \mathbb{R}^3\}
$$

Consider the existence of approximation function $F$, corresponding to element $f$ on the same space, $C([-1, +1])$. Let the approximation function has the form of

$$
F(A, x) = a_1 + a_2 \log (1 + a_3 x)
$$

for an element $A$ of a selected parameter space, $P$. When $\|a_3\| \geq 1$, $\|F(A, x)\|$ goes infinity so that the parameter $a_3$ satisfies

$$
-1 < a_3 < +1
$$

After selecting an approximation function $F$ as above, finding element $A^*$ for which $\|F - F(A^*)\|$ is minimum, gives solution of
Chebyshev problem. Such an element $A^*$ is called "best parameter" and $F(A^*,x)$ "best approximation" to $f$.

We can put approximation functions of the type

$$F(A,x) = a_1 + a_2 \log(1 + a_3 x)$$

into two groups:

1. **Constant approximation**

   Constant approximation is such approximation functions that correspond to parameters $A = (a_1,0,a_2)$ or $A = (a_1,a_2,0)$. Really in this case $F(A,x) = a_1$.

2. **Non-constant approximation**

   Now $a_2 \neq 0$ and $a_3 \neq 0$, that is $a_2 a_3 \neq 0$. In this case approximation function is evidently unique.

   **Lemma 1**: The difference between a constant approximation and another approximation has at most one zero in $[-1, +1]$.

   **Proof**: Constant approximation is $F(A,x) = a_1$ when $A$ has the form $A = (a_1, 0, a_2)$ or $A = (a_1, a_2, 0)$. Now, let non-constant another approximation function

   $$F(B,x) = b_1 + b_2 \log(1 + b_3 x)$$

   Due to the definition, $b_2 b_3 \neq 0$.

   Consider that

   $$d(x) = F(A,x) - F(B,x)$$

   has two zeros in $[-1, +1]$. According to Rolle theorem

   $$d'(x) = F'(A,x) - F'(B,x)$$

   has zero at least for one $x$ value. That is

   $$d'(x) = -\frac{b_2 b_3}{1 + b_3 x} = 0$$

   This implies $b_2 = 0$ or $b_3 = 0$. However, this is a contradiction to the assumption that $b_2 b_3 \neq 0$.

   **Lemma 2**: The difference between a non-constant approximation and a linear approximation has at most two zeros in $[-1, +1]$. 

Proof: Under the circumstances of \(-1 < a_3 < +1\), consider the difference between
\[
F(A,x) = a_1 + a_2 \log(1 + a_3 x) \quad \text{and} \quad a_4 + a_5 x
\]
Suppose \(d(x) = F(a,x) - a_4 - a_5 x\) has three zeros in \([-1, +1]\). Then derivative of \(d(x)\),
\[
d'(x) = \frac{a_2 a_3 - a_4 - a_5 a_6 x}{1 + a_3 x}
\]
has at most zeros in \([-1, +1]\).

For the approximation function, \(F(A,x) = a_1 + a_2 \log(1 + a_3 x)\), to be definite in \([-1, +1]\), \(1 + a_3 x > 0\) is required. Then the right hand side of
\[
(1 + a_3 x) \; d'(x) = a_2 a_3 - a_4 - a_5 a_6 x
\]
is a polynomial of first degree and has at most one zero. On the other hand if \(d'\) is identically zero then
\[
a_2 a_3 - a_4 = 0
\]
and
\[
a_5 a_3 = 0
\]
\(F(\cdot)\) is another non-constant approximation, so \(a_2 a_3 \neq 0\). Then \(a_3 = 0\). Inserting this value in the above equation we have \(a_2 a_3 = 0\). However, this a contradiction to the non-constant approximation, \(F(\cdot)\).

Lemma 3: The difference between a non-constant approximation and another approximation has at most two zeros in \([-1, +1]\).

Proof: Let \(F(\cdot)\) and \(F(B,\cdot)\) be two non-constant approximation functions.

Suppose \(d(x) = F(A,x) - F(B,x)\) has three zeros, so \(d'(x)\) has the form of
\[
d'(x) = F'(A,x) - F'(B,x) = \frac{(a_2 a_3 - b_2 b_3) + (a_2 a_3 - a_5 a_2) x}{(1 + a_3 x)(1 + b_3 x)}
\]
which must have at most two zeros. \(F(A,x)\) and \(F(B,x)\) to be definite in \([-1, +1]\) so that \(1 + a_3 x > 0\) and \(1 + b_3 x > 0\) are required. Then the right hand side of
(1 + a_2 x) (1 + b_2 x) \frac{d'(x)}{dx} = (a_2 a_3 - b_2 b_3) + (a_2 a_3 b_3 - a_1 b_3) x

is a polynomial of the first degree so that it has at most one zero and then \( d \) has at most two zeros.

On the other hand if \( d' \) is identical to zero, \( d \) must be constant. In that case \( d \) has zeros if and only if \( d' = 0 \). This is a contradiction. More clearly

\[ a_2 a_3 - b_2 b_3 = 0 \]

and

\[ a_1 b_3 (a_2 - b_2) = 0 \]

are required. Approximation functions are not constant, hence \( a_2 a_3 \neq 0 \) and \( b_2 b_3 \neq 0 \). From the second equation we find \( a_2 = b_2 \) and inserting it in the first equation we have \( a_1 = b_3 \) and \( d = a_2 - b_2 \). Here again if \( d \) has zeros which imply \( a_1 = b_3 \) then we get \( F(A,\cdot) = F(B,\cdot) \) which contradicts the assumption.

**Definition 1:** Define linear space \( D(A,\cdot,\cdot) \) formed by

\[ \frac{\partial F(A,\cdot)}{\partial a_i}, \text{ where } i = 1, 2, 3 \text{ and let the dimension be } d(A). \] Then \( d(A) \) evidently depends on \( A \).

If each non-zero element of linear space \( D(A,\cdot,\cdot) \) has at most \( d(A) - 1 \) zeros at element \( B \) of parameter space \( P \) then the space \( D(A,\cdot,\cdot) \) has “Classical HAAR” property.

A linear space that has the property of classical Haar is called Haar subspace.

**Lemma 4:** If \( D(A,\cdot,\cdot) \) correspond a constant approximation there exists a parameter \( A \) with a Haar subspace of dimension two.

**Proof:** Let \( A = (a_1, a_2, a_3) \), then it has continuous derivatives,

\[ \frac{\partial F(A, x)}{\partial a_i}, \quad \frac{\partial F(A, x)}{\partial a_2} = \log(1 + a_2 x), \quad \frac{\partial F(A, x)}{\partial a_2} = \frac{a_2 x}{1 + a_3 x} \]

Let \( B = (b_1, b_2, b_3) \), then an element of \( D(A, \cdot, \cdot) \) has the following form,

\[ D(A, B, x) = \sum_{i=1}^{3} b_i \frac{\partial F(A, x)}{\partial a_i} = b_1 + b_2 \log(1 + a_2 x) + b_3 \frac{a_2 x}{1 + a_3 x} \]
If we select the approximation function $F(A,\cdot)$ as constant and take $A=(a_1,0,a_2)$ then we have

$$D(A,B,x) = b_1 + b_2 \log(1+a_2x)$$

It is evidently seen that $D(A,B,x)$ is an element of linear space of two dimensions.

On the other hand, $D(A,B,x)$ has at most one zero in $[-1,+1]$ according to Lemma 1, under the condition that $D(A,B,x) \neq 0$.

In that case, $D(A,\cdot)$ is an "Haar subspace" of two dimensions for $A=(a_1,0,a_2)$.

**Lemma 5:** If $F(A,\cdot)$ is any non-constant approximation then $D(A,\cdot)$ is a Haar subspace of dimension 3.

**Proof:** Since the approximation function $F(A,\cdot)$ is non-constant $a_2$ and $a_3$ are non-zero and

$$D(A,B,x) = b_1 + b_2 \log(1+a_2x) + b_3 \frac{a_2x}{1+a_2x}$$

is clearly an element of vector space of dimension 3. This shows that $D(A,\cdot)$ is a linear vector space of dimension 3.

Let $D(A,B,x)$ be a non-zero element of $D(A,\cdot)$ then $B=(b_1,b_2,b_3) \neq 0$. Since

$$D'(A,B,x) = \frac{(b_2a_2 + b_3a_3) + b_2a_3^2x}{(1 + a_3x)^2}$$

has at most one zero in $[-1,+1]$ then $D(A,B,x)$ has at most two zeros. On the other hand since $D'(A,B,x) = 0$ then $b_2a_2 + b_3a_2 = 0$ and $b_2a_3^2 = 0$. Using $a_2 \neq 0$ and $a_3 \neq 0$ circumstances, we have $b_2=0$ and $b_3=0$. That is

$$D(A,B,x) = b_1$$

From the assumption $B=(b_1,b_2,b_3) \neq 0$ it is necessary to be $b_1 \neq 0$. In that case $D(A,\cdot)$ is a Haar subspace of dimension 3.

**Remark 1:** If $A$ corresponds to a constant approximation function, Lemma 1 shows that $d(A)=2$. Otherwise Lemma 3 gives $d(A)=3$. 
Now, to obtain a result of DE LA VALLEE-POUSSIN type which is useful in characterizing "near best approximation", let us consider a compact-Hausdorff space, $X$ and prove some theorems.

Let us consider a compact Hausdorff space $X$, and a set $C(X)$ of all continuous functions on $X$. If $P$ be a parameter space and $f$ be any element of $C(X)$ then $S(A,B;x)$ is defined such as

$$S(A,B;x) = (F(A,x) - f(x)) (F(A,x) - F(B,x))$$

where $A$ and $B$ are elements of $P$. Now, let us prove that

$$\rho(f) = \inf \{ \| F(A,.) - f \| : A \in P \}$$

has a sublimit.

**Theorem 1:** Let $A$ be an element of parameter space, $P$. If for each element, $B$, of $P$, there is a closed subset, $K$, of $X$ such that

$$\min \{ S(A,B;x) : x \in K \} \leq 0$$

then

$$\rho(f) \geq \min \{ |F(A,x) - f(x)| : x \in K \} = \sigma$$

**Proof:** Suppose $\rho(f) < \sigma$ then

$$\rho(f) < \| F(B,.) - f \| < \sigma$$

such that there exists an element, $B$, of $P$. Hence for the elements $x$ of $K$

$$|F(A,x) - f(x)| - |F(B,x) - f(x)| > 0$$

and

$$S(A,B,x) = |F(A,x) - f(x)|^2 - (F(A,x) - f(x))(F(B,x) - f(x))$$

$$\geq |F(A,x) - f(x)| (|F(A,x) - f(x)| - |F(B,x) - f(x)|) > 0$$

This contradicts the hypothesis.

**Definition 2:** For a $g$ element of space $C([-1, +1])$ if there exist

$$|g(x_i)| = \| g \|, g(x_i) = (-1)^i g(x_i); \ (i = 1, 2, ..., d(A))$$

and point set $\{ x_1, x_2, ..., x_{d(A)+1} \}$ such that $-1 \leq x_1 < ... < x_{d(A)+1} \leq +1$ then $g$ function alternates $d(A)$ times.

**Theorem 2:** If approximation function $F$ has property $(Z)$ at $A$ and for an element $f$ of $C([-1, +1])$, $F(A,.) - f$ alternates on $\{ x_1, x_2, ..., x_{d(A)+1} \}$ then there exists property
\[ \rho(f) \geq \min \{|F(A,x_k) - f(x_k)| : 1 \leq k \leq d(A) + 1\} \]

**Proof:** Since the function \( F(A,\cdot) - f \) changes alternatively on \( \{x_1, x_2, \ldots, x_{d(A)+1}\} \), there exists the property

\[ \text{Sgn} \ (F(A,x_j) - f(x_j)) = - \text{Sgn} \ (F(A,x_{j+1}) - f(x_{j+1})) \]  

where, \( j = 1, 2, \ldots, d(A) \)

Let \( K \) in theorem 1 as \( K = \{x_k; 1 \leq k \leq d(A) + 1\} \) then one gets

\[ \rho(f) \geq \min \{|F(A,x_k) - f(x_k)| : 1 \leq k \leq d(A) + 1\} \]

In that case at least for an \( x_p \in K \), one gets

\[ S(A,B,x_p) = (F(A,x_p) - f(x_p)) - (F(A,x_p) - F(B,x_p)) \leq 0 \]

Otherwise \( F(A,\cdot) - F(B,\cdot) \) has \( d(A) + 1 \) zeros in \([-1, +1]\) according to the property (1). This contradicts the hypothesis that \( F(A,\cdot) \) has property (Z) at \( A \).

**Definition 3:** Approximation function \( F(A,\cdot) \) has the property of local Haar space, with null points of degree \( d(A) \) at \( A \), if the following conditions are fulfilled:

(I) Approximation function \( F(A,\cdot) \) has continuous partial derivatives for each \( i, i = 1, 2, \ldots, n \).

(II) Setting

\[ D(A,B,x) = (B, \nabla F(A,x)) = \sum_{i=1}^{n} b_i \frac{\partial F(A,x)}{\partial a_i} \]

we have

\[ F(A+B,x) - F(A,x) = D(A,B,x) + R(A,B,x) \]

and when \( \|B\| \) is sufficiently small

\[ R(A,B,x) = O(\|B\|) \]

(III) There exists a neighbourhood of element \( A \) which is contained in \( P \).

(IV) Linear space \( D(A,\cdot) \) is a Haar subspace of dimension \( d(A) \) in \([-1, +1]\).

**Remark 2:** Approximation function \( F \) has local Haar space condition, only when \( D(A,\cdot) \) obeys classical Haar condition.

**Theorem 3:** If approximation function \( F \) has the local property with null points of degree \( d(A) \) at \( A \) and function \( f \) be
an element of space $C([-1, +1])$, and $F(A,.)$ be the best approximation to $f$, then function $F(A,.) - f$ alternates $d(A)$ times.

Proof: Let $F$ be the best approximation to $f$, then set of extreme points of $F(A,x)-f(x)$,

$$M_A = \{x / x \in [-1, +1] : \|F(A,.) - f\| = |F(A,x) - f(x)|\}$$

has at least $d(A)+1$ elements.

Under the above conditions there exist some points which hold $-1 \leq x_i < \ldots x_{d(A)+1} < +1$ and set $\{x_1, x_2, \ldots, x_{d(A)+1}\}$ is an alternant of $F(A,.) - f$. Otherwise there would be found a natural number, $m$, and so we can separate $[-1, +1]$ into $m+1$ subintervals such that each interval contains an extreme point and $F(A,x)-f(x)$ has same sign in these intervals.

The set of extreme points of $F(A,x)-f(x)$, has $d(A)+1$ elements, hence, for $k=1,2,\ldots, d(A)$, a non-zero element $B$ of parameter space $d'$ can be found [2] such that

$$(B, \nabla F(A,x_k)) = \sum_{i=1}^{d} b_i \frac{\partial F(A,x_k)}{\partial a_i} - F(A,x_k) - f(x_k)$$

and so for all extreme points, $x$,

$$(F(A,x)-f(x))(B, \nabla F(A,x)) = |F(A,x)-f(x)|^2$$

and then

$$\text{Sgn}(B, \nabla F(A,x)) = \text{Sgn}(F(A,x) - f(x))$$

This result contradicts the hypothesis of the best approximation function $F$ to $f$.

Meinardus and Schwedt ([2] theorem 9) showed that a set $M_A$ of extreme points, has at most $d(A)+1$ points in $[-1, +1]$.

Opposition of the Theorem 2 is correct, provided the above conditions are taken into account.

Now, combining Theorem 2 and Theorem 3 one can get the following result:

Theorem 4: If $F(A,.)$ has property (Z) at $A$ and local Haar property with null points of degree $d(A)$, then $F(A,.)$ be the best to $f$ if and only if $F(A,.)-f$ alternates $d(A)$ times.
Theorem 5: If $F(A, \cdot)$ satisfies the condition of Theorem 4, and $F(A, \cdot)$ is best, then it is a unique best approximation.

Proof: Suppose, $F(A, \cdot)$ and $F(B, \cdot)$ are two approximation functions. We can take $d(A) \leq d(B)$, without violating the generality.

Let set of extreme points of $F(A, \cdot) - f$ be $\{x_1, x_2, \ldots, x_{d(A)+1}\}$ ($k = 1, 2, \ldots, d(A)+1$). According to Theorem 3, the set $\{x_1, x_2, \ldots, x_{d(A)+1}\}$ is an alternant of $F(A, \cdot) - f$. Then we have

$$F(A, x_{j+1}) - f(x_{j+1}) = -(F(A, x_j) - f(x_j))$$

where, $j = 1, 2, \ldots, d(A)$. Hence using Equation 1 we get inequalities system

$$F(A, x_1) - F(B, x_1) \leq 0$$
$$F(A, x_2) - F(B, x_2) \geq 0$$

or

$$\begin{align*}
F(A, x_1) - F(B, x_1) & \geq 0 \\
F(A, x_2) - F(B, x_2) & \leq 0
\end{align*}$$

It is sufficient to investigate the first part,

$$\begin{align*}
F(A, x_1) - F(B, x_1) & \leq 0 \\
F(A, x_2) - F(B, x_2) & \geq 0
\end{align*}$$

If the inequalities had been certain, $F(A, \cdot) - F(B, \cdot)$ would have had $d(A) + 1$ definite null points and from the Haar condition we would have gotten result

$$F(., \cdot) = F(B, \cdot)$$

On the other hand, if the inequalities had been correct for a $k_0$, we would have gotten

$$F(A, x_{k_0}) - F(B, x_{k_0}) \neq 0$$

$$\text{Sgn} \ (F(A, x_{k_0}) - F(B, x_{k_0})) = (-1)^{k_0}$$

However, if $(F(., \cdot)$ and $F(B, \cdot)$ are two approximation functions and if we take

$$A(t) = (1-t) A + t B$$
$$B(t) = (1-t) B + t A$$
then \( F(A(t),.) \) and \( F(B(t),.) \) are also approximation functions. If we denote \( \delta = B - A \) in

\[
B(t) = B - t \ (B - A)
\]

we get

\[
B(t) = B - t \ \delta
\]

where, parameter \( \delta \) is an element of space \( p \).

Since \( D(B,.) \) satisfies Haar condition, each non-zero element of \( D(B,.) \) has at most \( d(A)-1 \) null points at element \( \delta \) of parameter space \( P \). So \( F(B,.) \) have local Haar property.

Using property (II) of local Haar condition in \( F(B,x)-F(B-t\delta,x) \) we get

\[
F(B,x) - F(B-t\delta,x) = tD(B,\delta,x) + R(B,\delta,x)
\]

and adding the approximation function \( F(A,.\) to the each side of this equation and denoting \( R(B,\delta, x) = 0 \) \( \) we find

\[
F(A,x) - F(B-t\delta,x) = F(A,x) - F(B,x) + tD(B,\delta,x) + 0(t)
\]

We get the following system, for \( t > 0 \),

\[
F(A,x_1) - F(B-t\delta,x_1) < 0
\]

\[
F(A,x_2) - F(B-t\delta,x_2) > 0
\]

\[\cdots\cdots\cdots\cdots\cdots\cdots\cdots\]

Thus \( F(A,.) - F(B-t\delta,.) \) has at least \( d(A) \) null points in \([-1, +1]\) and when \( t \) is approaching to zero we get

\[
F(A,.) = F(,.)
\]

ÖZET

Chebyshev yaklaşımı üzerine daha önce yapılan çalışmalar aydınlatılmış, \([9,2]\) üzerine en iyi Chebyshev yaklaşımının \( A+B^* \log (1+CX) \) olduğu ispatlanmış ve yeni kavramlar yardımcıla konu genelleştirilmiştir.

REFERENCES

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