On Almost-Continuity And Almost-A Continuity
Of Real Functions

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SUMMARY

The purpose of this note is to give some new concepts of continuity for real functions and
to investigate the relations between concepts of continuity.

1. INTRODUCTION

Let $A = (a_{nk})$ be an infinite matrix of real numbers and $x = (x_k)$ be
a sequence of real numbers. The sequence $((Ax)_n)$ defined by

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k$$

(1)
is called the $A$-transform of $x$ whenever the above series converges for
$n = 1, 2, \ldots$. The sequence $x$ is said to be $A$-summable to $x_0$ if the
sequence $((Ax)_n)$ converges to $x_0$. $A$ is called conservative if $x \in c$ implies $((Ax)_n \in c$, where $c$ is the linear space of convergent sequences. $A$ is
called regular if it is conservative and preserves the limit of each convergent
sequence. $A$ is called strongly regular if $A$ is regular and

$$\lim_n \sum_{k=1}^{\infty} |a_{nk} - a_{n,k+1}| = 0$$

(2)

[3]. Throughout this study $R$ stands for real numbers and $N$ denotes
the set of positive integers.

2. Definitions.

Let $m$ denote the linear space of bounded sequences.

A sequence $x \in m$ is said to be almost convergent and $s$ is called its generalized
limit if each Banach limit of $x$ is $s$. [3]. The class $F$ of almost con-
vergent sequences was characterized by G. G. Lorentz [3], who proved that a sequence \( x = (x_k) \) is almost convergent if and only if

\[
\lim_p \frac{x_n + x_{n+1} + \cdots + x_{n+p-1}}{p} = s
\]  

(uniformly in \( n \)). We shall write \( F\text{-}\lim x = s \) or \( \text{Lim } x = s \), shortly. We denote by \( Lx \) the following sequence

\[
\left( \frac{1}{p} \sum_{j=n}^{n+p-1} x_j \right).
\]

If the method \( A \) sums all almost convergent sequences then \( A \) is called strongly regular [3]. It is clear that a convergent sequence is almost convergent and its limit and generalized limit are identical.

We shall now speak of some basic concepts. Let \( X, Y \) be topological spaces. Then \( f: X \rightarrow Y \) is called continuous on \( X \) if and only if the inverse image of every open set in \( Y \) is open in \( X \) and \( f \) is called sequentially continuous at a point \( x_0 \in X \) if and only if for every sequence \( x_n \rightarrow x_0 \) (in \( X \)) we have \( f(x_n) \rightarrow f(x_0) \) (in \( Y \)). It is known that if \( f: X \rightarrow Y \) is continuous on \( X \), then \( f \) is sequentially continuous on \( X \), but not conversely in general. Furthermore, if \( X, Y \) are metric spaces, then the sequentially continuity on \( X \) implies continuity on \( X \) [4]. Thus the concepts of sequential continuity and continuity coincide for \( R \), since \( R \) is a metric space with the usual modulus metric.

A function \( f: R \rightarrow R \) is called \( c \)-continuous at the point \( x_0 \in R \) if

\[
(c, 1) - \lim f(x_n) = f(x_0) \quad \text{whenever} \quad (c, 1) - \lim x_n = x_0 [6],
\]

where \( (c, 1) \) is the first Cesàro mean and \( (c, 1) - \lim x_n = x_0 \) means that

\[
\frac{x_1 + x_2 + \cdots + x_n}{n} \rightarrow x_0 \quad (n \rightarrow \infty)
\]

Similarly, \( A \)-continuity of \( f \) was defined by Jozef Antoni-Tibor Salat [1].

We shall give some new additional definitions:

**Definition (2.1).** Let \( x = (x_n) \) be a sequence in \( R \). We shall say that a function \( f: R \rightarrow R \) is almost continuous at the point \( x_0 \in R \) if \( F\text{-}\lim (f(x)) = f(x_0) \) whenever \( F\text{-}\lim x = x_0 \).
Definition (2.2). Let $A = (a_{nk})$ be a regular matrix of real numbers and $x = (x_n)$ be a sequence in $\mathbb{R}$. We shall say that a function $f: \mathbb{R} \to \mathbb{R}$ is $A$-almost continuous at $x_0 \in \mathbb{R}$ if $A$-lim \ (Lf(x)) = f (x_0)$ whenever $A$-lim \ (Lx) = x_0.$

Definition (2.3). Let the matrix $A = (a_{nk})$ and the sequence $x = (x_n)$ be as the definition (2.2). We shall say that a function $f: \mathbb{R} \to \mathbb{R}$ is almost $A$-continuous at $x_0 \in \mathbb{R}$ if $F$-lim \ (A (f(x))) = f (x_0)$ whenever $F$-lim \ (Ax) = x_0.$

In the case of $A$ is a unit matrix the definitions (2.2) and (2.3) are equivalent.

3. Relations between the concepts of continuity.

Theorem (3.1). If a function $f: \mathbb{R} \to \mathbb{R}$ is $A$-almost continuous at $x_0 \in \mathbb{R}$ then $f$ is almost continuous at the same point.

Proof. Let $x = (x_n)$ be a sequence in $\mathbb{R}$ such that $Lx$ converges to $x_0$. Since $f$ is $A$-almost continuous at $x_0 \in \mathbb{R}$

$A$-lim \ (Lx) = x_0$ implies $A$-lim \ (Lf(x)) = f (x_0),

and so,

Lim \ $x = x_0$ implies $A$-lim \ (Lx) = x_0 implies $A$-lim \ (Lf(x)) = f(x_0).

Hence, $Lim \ x = x_0$ implies $A$-lim \ (Lf(x)) = f (x_0),$

that is, we have $A$-lim \ (Lf(x)) = f (x_0) for very sequence $Lx$ converging to $x_0$. On the other hand, every subsequence of $Lx$ converges to $x_0$ since $Lx$ converges to $x_0$. It is easy to see that to each subsequence of $Lf (x)$ there corresponds a subsequence of $Lx$ which is convergent to $x_0$. Therefore, $A$-soms every subsequence of $Lf(x)$. Hence the sequence $Lf(x)$ is convergent [2]. Moreover the sequence $Lf(x)$ must converge to $f (x_0)$ since $A$ is regular and $A$-lim \ (Lf(x)) = f (x_0)$ This completes the proof.

Theorem (3.2). Let $f: \mathbb{R} \to \mathbb{R}$ be an almost continuous function at $x_0 \in \mathbb{R}.$ Then $f$ is continuous at $x_0$ if and only if

$$f (x_{n+1}) - f (x_n) \to o \ (n \to \infty) \quad (5)$$
for each sequence \( x = (x_n) \) converging to \( x_0 \).

**Proof.** Necessity. Let \( f \) be continuous at \( x_0 \in \mathbb{R} \). Then,

\[ x_n \to x_0 \quad (n \to \infty) \implies f(x_n) \to f(x_0) \quad (n \to \infty). \]

Hence, for every number \( \varepsilon > 0 \) there exists \( n_0(\varepsilon) \) such that

\[ |f(x_n) - f(x_0)| < \frac{\varepsilon}{2} \]

for each \( n > n_0(\varepsilon) \). Therefore, for \( n > n_0(\varepsilon) \) we have

\[ |f(x_{n+1}) - f(x_n)| \leq |f(x_{n+1}) - f(x_0)| + |f(x_n) - f(x_0)| < \varepsilon. \]

Sufficiency. Let the sequence \( x = (x_n) \) converge to \( x_0 \) and \( f \) be an almost continuous function at \( x_0 \in \mathbb{R} \). Then for any number \( \varepsilon > 0 \), we can choose a number \( p \) large enough such that

\[ \left| \frac{1}{p} \left( f(x_n) + f(x_{n+1}) + \ldots + f(x_{n+p-1}) \right) - f(x_0) \right| < \frac{\varepsilon}{2} \quad (6) \]

for all \( n \in \mathbb{N} \).

Let us take \( \varepsilon_1 = \frac{\varepsilon}{p-1}, \) (\( p > 1 \)). By (5), for the number \( \varepsilon_1 > 0 \) we select a number \( n_0 \) so large that

\[ |f(x_n) - f(x_{n+1})| < \varepsilon_1 \]

for all \( n > n_0 \). Therefore, for \( n > n_0 \) we get

\[ |f(x_n) - f(x_{n+p-1})| \leq (p - 1) \varepsilon_1. \quad (7) \]

Let \( \max(n_0, p) = M \). By (6) and (7), for \( n > M \) we have

\[
|f(x_n) - f(x_0)| \leq |f(x_n) - \frac{f(x_n) + \ldots + f(x_{n+p-1})}{p} | \\
+ \left| \frac{f(x_n) + \ldots + f(x_{n+p-1})}{p} - f(x_0) \right| \\
\leq \frac{1}{p} \left| p \cdot f(x_n) - \sum_{j=n}^{n+p-1} f(x_j) \right| + \frac{\varepsilon}{2} \\
\leq \frac{1}{p} \sum_{j=n}^{n+p-1} |f(x_n) - f(x_j)| + \frac{\varepsilon}{2}
\]
\[ \leq \frac{1}{p} \left( 1 + 2 + \ldots + (p - 1) \right) \varepsilon_1 + \frac{\varepsilon}{2} = \varepsilon \]

This completes the proof.

In a recent paper, we have defined the new methods of summability by a suitable rearrangement of the elements on each row of a given matrix summability method [5]. In connection with this we can give the following:

**Theorem** (3.3). Let \( A = (a_{nk}) \) be a strongly regular matrix and \( f: R \rightarrow R \) be a function such that the sequence \( f(x) \) is bounded whenever \( x = (x_k) \) is bounded. Then the concepts of the A-continuity and the \( A_\pi \)-continuity corresponding to those permutation functions each of which has a symmetrical mapping (see, definition in [5]) on disjoint blocks of the positive integers, are equivalent.

**Proof.** We showed in theorem 2.1 [5] that for every bounded sequence \( x = (x_k) \) we have

\[ \lim_{n} |(Ax)_n - (A_\pi x)_n| = 0. \]  

(8)

Let \( A_\pi \)-lim \( x_n = x_0 \) and \( f \) be \( A \)-continuous at \( x_0 \in R \). We shall show that \( A_\pi \)-lim \( f(x_n) = f(x_0) \). Since \( f \) is \( A \)-continuous at \( x_0 \in R \) we have

\( A \)-lim \( x_n = x_0 \) implies \( A \)-lim \( f(x_n) = f(x_0) \).

By (8) and since \( A \)-lim \( f(x_n) = f(x_0) \), we get \( A_\pi \)-lim \( f(x_n) = f(x_0) \). Hence, \( f \) is \( A \)-continuous at \( x_0 \in R \). In the same way one can prove that \( f \) is \( A_\pi \)-continuous at \( x_0 \in R \) if the function \( f \) is \( A_\pi \)-continuous at \( x_0 \in R \). This completes the proof.

**ÖZET**

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**REFERENCES**


