Properties of 2-Dimensional Ruled Surfaces In The Euclidean n-Space E^n And Massey’s Theorem

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Properties of 2-Dimensional Ruled Surfaces In The Euclidean
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ABSTRACT

In this paper we find new characteristic properties for 2-dimensional ruled surfaces $M$
in $E^n$ and we give the sufficient and necessary conditions for which the ruled surface $M$ is to be
total geodesic. In addition, the Massey's theorem which is well-known for the ruled surfaces in
the Euclidean 3-space, [3], was generalized for the ruled surfaces in $E^n$.

I. INTRODUCTION

We will assume throughout this paper that all manifolds, maps, vector fields, etc. . . . are differentiable of class $C^\infty$. Consider a general submanifold $M$ of the Euclidean n-space $E^n$. Suppose that $\mathcal{D}$ is the Riemann connection of $E^n$, while $D$ is the Riemann connection of $M$. Then, if $X$ and $Y$ are the vector fields of $M$ and if $V$ is the second fundamental form of $M$, we have by decomposing $\mathcal{D}_X Y$ in a tangential and a normal component

(I.1) \[ \mathcal{D}_X Y = D_X Y + V(X, Y). \]

The equation (I.1) is called Gauss equation.

If $\xi$ is any normal vector field on $M$, we find the Weingarten equation by decomposing $\mathcal{D}_X \xi$ in a tangential component and a normal component

(I.2) \[ \mathcal{D}_X \xi = - A_\xi(X) + \nabla_X \xi. \]

$A_\xi$ determines at each point a self-adjoint linear map and $\nabla_X$ is a metric connection in the normal bundle $\frac{\xi}{\xi}(M)$. We use the same
notation $A_{\xi}$ for the linear map and the matrix of the linear map. A normal vector field $\xi$ is called parallel in the normal bundle $\mathcal{N}(M)$ if we have $D_X \xi = 0$ for each $X \in \mathcal{X}(M)$. If $\eta$ is a normal unit vector at the point $p \in M$, then

$$(1.3) \quad G(p, \eta) = \det A_\eta$$

is the Lipschitz-Killing curvature of $M$ at $p$ in the direction $\eta$.

Suppose that $X$ and $Y$ are vector fields on $M$, while $\xi$ is a normal vector field, then, if the standard metric tensor of $\mathbb{E}^n$ is denoted by $<,>$

$$(1.4) \quad X <Y, \xi> = <D_X Y, \xi> + <Y, D_X \xi> = 0$$

or

$$<V(X,Y), \xi> = <Y, A_\xi(X)>.$$  

If $\xi_1, \xi_2, \ldots, \xi_{n-2}$ constitute an orthonormal base field of the normal bundle $\mathcal{N}(M)$, then we set

$$(1.5) \quad <V(X,Y), \xi_i> = V_i(X,Y)$$

or

$$V(X,Y) = \sum_{i=1}^{n-2} V_i(X,Y) \xi_i.$$  

The mean curvature vector $H$ of $M$ at the point $p$ is given by

$$(1.6) \quad H = \sum_{i=1}^{n-2} \frac{\text{tr} A_{\xi_i}}{2} \xi_i.$$  

$||H||$ is the mean curvature. If $H=0$ at each point $p$ of $M$, then $M$ is said to be minimal.

II. 2-DIMENSIONAL RULED SURFACES IN THE EUCLIDEAN $n$-SPACE $\mathbb{E}^n$

Suppose that the base curve $r(s)$ of the 2-dimensional ruled surface $M$ in $\mathbb{E}^n$ is an orthogonal trajectory of the generators, which have the direction of the unit vector $e(s)$; then $M$ can locally be represented by
\[ \varphi(s,l) = r(s) + le(s). \]

**Definition II.1:** Let \( M \) be a 2-ruled surface in \( E^n \) and \( V \) be the second fundamental form of \( M \). If \( V(X,X) = 0 \) for all \( X \in \mathcal{X}(M) \), then \( X \) is called an asymptotic vector field on \( M \).

**Theorem II.1:** Let \( M \) be a 2-dimensional ruled surface in \( E^n \). Then the generators of \( M \) are asymptotics and geodesics of \( M \).

Proof: Since the generators are the geodesics of \( E^n \), we have \( \mathbf{D}_e\mathbf{e} = 0 \).

If we set this in the Gauss equation, we get

\[ \mathbf{D}_e\mathbf{e} + V(e,e) = 0 \text{ or } \mathbf{D}_e\mathbf{e} = -V(e,e). \]

Since \( \mathbf{D}_e\mathbf{e} \in \mathcal{X}_n(M) \) and \( V(e,e) \in \mathcal{X}_n(M) \) we find \( \mathbf{D}_e\mathbf{e} = 0 \) and \( V(e,e) = 0 \).

Therefore the generators of \( M \) are the asymptotics and geodesics of \( M \).

Suppose that \( \{e_1, e\} \) is an orthonormal base field of the tangential bundle \( \mathcal{X}_n(M) \) and \( \{\xi_1, \xi_2, \ldots, \xi_{n-2}\} \) is an orthonormal base field of the normal bundle \( \mathcal{X}_n(M) \). Then we have the following equations.

\[ \mathbf{D}_e\xi_j = a_{j1} e + a_{j2} e_1 + \sum_{i=1}^{n-2} b_{ij} \xi_i \]

(II.1)

\[ \mathbf{D}_e\xi_j = a_{j1} e + a_{j2} e_1 + \sum_{i=1}^{n-2} b_{ij} \xi_i, \quad 1 \leq j \leq n-2. \]

From these equations we observe that

\[ A_{\xi_j} = - \begin{bmatrix} a_{j1} & a_{j2} \\ a_{j2} & a_{j2} \end{bmatrix}, \quad 1 \leq j \leq n-2. \]

Since \( \mathbf{D}_e\xi_j \) and \( \mathbf{D}_e\xi_j \) are orthogonal to \( \xi_j \), we have \( b_{ij} = 0 \).

On the other hand, \( a_{j1} = <\mathbf{D}_e\xi_j, e> = -<\xi_j, \mathbf{D}_e\mathbf{e}> \) and \( \mathbf{D}_e\mathbf{e} = 0 \), thus we find \( a_{j1} = 0, 1 \leq j \leq n-2. \) We also have
(II.2) \[ a_{12} = \langle D_2 \xi_j, e_1 \rangle = - \langle \xi_j, D_2 e_1 \rangle \]
and
(II.3) \[ \langle D_0 e_1, e \rangle = - \langle e_1, D_0 e \rangle = 0 \]
while
(II.4) \[ \langle D_0 e_1, e_1 \rangle = - \langle e_1, D_0 e_1 \rangle = 0. \]

From (II.3) and (II.4) we observe that \[ D_0 e_1 \in \mathcal{H} (M) \] or \[ D_0 e_1 = V(e,e_1), \]
because of (II.2) we have
(II.5) \[ D_0 e_1 = V(e,e_1) = \sum_{j=1}^{n-2} \langle \xi_j, D_0 e_1 \rangle = - \sum_{j=1}^{n-2} a_{12} \xi_j. \]

Because of (I.4) and (II.1) we find
(II.6) \[ a_{22} = \langle D_0 \xi_j, e_1 \rangle = - \langle A \xi_j (e_1), e_1 \rangle = - \langle V(e_1, e_1), \xi_j \rangle \]
and
(II.7) \[ \text{tr} A \xi_j = - a_{22} = \langle V(e_1, e_1), \xi_j \rangle, \quad 1 \leq j \leq n-2. \]

**THEOREM II.2:** Let \( M \) be a 2-ruled surface in \( E^n \) and \{e_1, e\} be the orthonormal base field of \( M \). Then the Gauss curvature \( G \) is given by
(II.8) \[ G = - \langle D_e e_1, D_{e_1} e_1 \rangle \]
where \( \bar{D} \) denotes the Riemann connection of \( E^n \). [4].

By using Theorem II.2 and (II.5) we find
(II.8) \[ G = - \sum_{j=1}^{n-2} (a_{12})^2. \]

On the other hand, because of the expressions stated in (I.6) and (II.7) we have
(II.9) \[ H = \sum_{j=1}^{n-2} \frac{\langle V(e_1, e_1), \xi_j \rangle \xi_j}{2} = 1/2 \ V(e_1, e_1). \]

**DEFINITION II.2:** Let \( M \) be a 2-ruled surface in \( E^n \). If the tangent planes of \( M \) are constant along the generators of \( M \), \( M \) is called **developable**, [2].
DEFINITION II.3: Let $M$ be a 2-dimensional ruled surface in $E^n$ and $V$ be a second fundamental form of $M$. If

$$V(X, Y) = 0$$

for all $X, Y \in \mathcal{X}(M)$, then $M$ is called \textit{totally geodesic}, [1].

THEOREM II.3: A 2-ruled surface $M$ in $E^n$ is developable and minimal iff $M$ is total geodesic.

Proof: We assume that $M$ is developable and minimal. If $X, Y \in \mathcal{X}(M)$, we have $X = ae + be_1$ and $Y = ce + de_1$. Therefore we get

$$(II.10) \quad V(X, Y) = acV(e, e) + (ad + bc)V(e, e_1) + bdV(e_1, e_1).$$

Because of Theorem II.1 and minimality of $M$ we have $V(e, e) = 0$ and $V(e_1, e_1) = 0$. Moreover, since $M$ is developable $\nabla e_1 = 0$. Thus we can write $V(e, e_1) = 0$ and $V(X, Y) = 0$ for all $X, Y \in \mathcal{X}(M)$.

Now, suppose that $V(X, Y) = 0, \forall X, Y \in \mathcal{X}(M)$. Then we have $V(e, e) = 0, V(e, e_1) = 0$ and $V(e_1, e_1) = 0$. Because of Theorem II.1 we have

$$\langle \nabla e_1, e \rangle = 0 \text{ and } \langle \nabla e_1, e_1 \rangle = 0.$$

This means that $\nabla e_1$ is a normal vector field or $\nabla e_1 = V(e, e_1)$.

Therefore we have $\nabla e_1 = 0$. This implies that $M$ is developable and $V(e_1, e_1) = 0$ implies that $M$ is minimal.

That completes the proof of the theorem.

III. THE MASSEY'S THEOREM FOR 2-DIMENSIONAL RULED SURFACES IN THE EUCLIDEAN n-SPACE $E^n$

Consider a 2-dimensional ruled surface $M$ in $E^n$ and the unit vector field $e$ of the generator, then the orthonormal base field $\{e_1, e\}$ of the tangential bundle of $M$ at each point $p$ of $M$ and the orthonormal base field $\{\xi_1, \xi_2, \ldots, \xi_{n-2}\}$ of the normal bundle of $M$ at each point $p$ of $M$ constitute an orthonormal base field of $E^n$ at each point $p$ of $E^n$.

On the other hand, we have the equations of covariant derivative of the orthonormal base field $\{e_1, e, \xi_1, \xi_2, \ldots, \xi_{n-2}\}$ of $E^n$, in matrix form, as follows:
Now, we would like to generalize the Massey's theorem, which is well-known for the ruled surfaces in $E^3$, [3], to the ruled surfaces in the Euclidean $n$-space $E^n$.

**THEOREM III.1:** Let, $M$ be a 2-dimensional ruled surface in $E^n$, \{$e_1, e\}$ be an orthonormal base field of the tangential bundle $\frac{\mathcal{T}}{\mathcal{N}}(M)$ and $r(s)$ be an orthogonal trajectory of the generators of $M$. Then the following propositions are equivalent.

(i) $M$ is developable.

(ii) The Lipschitz-Killing curvature

$$G(p, \xi_j) = 0, \ 1 \leq j \leq n-2.$$  

(iii) The Gauss curvature $G = 0$.

(iv) In the equation (III.1), $c_{2k} = 0, \ 3 \leq k \leq n$.

(v) $A_{\xi_j}(e) = 0$.

(vi) $D_{e_1} e \in \frac{\mathcal{T}}{\mathcal{N}}(M)$.

**Proof:** (i) $\Rightarrow$ (ii): We assume that $M$ is developable. Since $a_{11} = 0$, in (II.1), $1 \leq j \leq n-2$, the Lipschitz-Killing curvature at point $p$ in the direction of $\xi_j$ is given by

$$G(p, \xi_j) = -(a_{12}(p))^2 = 0, \ 1 \leq j \leq n-2.$$  

Because of (II.5) and since $M$ is developable we have

$$D_{e_1}e_1 = -\sum_{j=1}^{n-2} (a_{12}) \xi_j = 0.$$  

So we find $G(p, \xi_j) = 0, \ 1 \leq j \leq n-2$.

(ii) $\Rightarrow$ (iii): Let $G(p, \xi_j) = 0, \ 1 \leq j \leq n-2$. Since we have, [4],
\[ G(p) = \sum_{j=1}^{n-2} G(p, \xi_j), \quad \forall p \in \mathbb{M} \]

we observe that \( G = 0, \forall p \in \mathbb{M} \).

(iii) \( \Rightarrow \) (iv): Suppose that \( G = 0, \forall p \in \mathbb{M} \). Then, because of (II.8) we have \( a_{12} = 0, 1 \leq j \leq n-2 \). So \( \bar{D}_{e_1} \xi_j \) has no component in the direction \( e \). Hence we observe that \( c_{2k} = 0, 3 \leq k \leq n \), in the equation (III.1).

(iv) \( \Rightarrow \) (v): Suppose \( c_{2k} = 0, 3 \leq k \leq n \), in the equation (III.1). This shows that \( \bar{D}_{e_1} \xi_j \) has no component in the direction \( e \). Thus we have, in the equation (II.1), \( a_{12} = 0, 1 \leq j \leq n-2 \).

Moreover, since \( a_{11} = \langle \bar{D}_{e_1} \xi_j, e \rangle = - \langle \xi_j, \bar{D}_e e \rangle = 0 \) and because of the Weingarten equation we find
\[ A_{\xi_j}(e) = 0, \quad 1 \leq j \leq n-2. \]

(v) \( \Rightarrow \) (vi): Let \( A_{\xi_j}(e) = 0 \). Then, from the Weingarten equation, we have \( a_{11} = 0, a_{12} = 0, 1 \leq j \leq n-2 \). Moreover, since \( \langle e, \xi_j \rangle = 0 \) implies \( \langle \bar{D}_{e_1} e_1, \xi_j \rangle = - \langle e, \bar{D}_{e_1} \xi_j \rangle = - a_{12} \), we find
\[ \langle \bar{D}_{e_1} e, \xi_j \rangle = 0. \]

So we get
\[ \bar{D}_{e_1} e \in \mathcal{X}(M). \]

(vi) \( \Rightarrow \) (i): Let \( \bar{D}_{e_1} e \in \mathcal{X}(M) \). Then \( \langle \bar{D}_{e_1} e, \xi_j \rangle = a_{12} = 0, 1 \leq j \leq n-2 \). On the other hand, \( \langle e_1, e_1 \rangle = 1 \) implies that \( \langle \bar{D}_e e_1, e_1 \rangle = 0 \) and \( \langle e_1, e \rangle = 0 \) implies that \( \langle \bar{D}_e e_1, e \rangle = 0 \). Thus \( \bar{D}_e e_1 \in \mathcal{X}(M) \).

Because of (II.5) and since \( a_{12} = 0, 1 \leq j \leq n-2 \), we write that \( \bar{D}_e e_1 = 0 \).

This means the tangent planes of \( M \) are constant along the generator \( e \) of \( M \), i.e. \( M \) is developale.

COROLLARY III.2: Let \( M \) be a 2-dimensional ruled surface in \( \mathbb{E}^n \) with a Gauss curvature being zero. If \( M \) is minimal, then \( c_{sk} = 0 \), \( 1 \leq s \leq 2, 3 \leq k \leq n \).

Proof: Let \( M \) be minimal. Then from the equation (II.9), we have \( V(e_1, e_1) = 0 \). If this result is set in the Gauss equation, we find
\[ D_{e_1} e_1 = D_{e_1} e_1. \]

This means that \( D_{e_1} e_1 \) has no component in \( \frac{1}{N} (M) \). Therefore we have

\[ c_{1k} = 0, \ 3 \leq k \leq n, \]

in the equation (III.1). On the other hand, since \( G = 0 \), by hypothesis, and from the Theorem III.1, we know that \( c_{2k} = 0, \ 3 \leq k \leq n \). If we consider this together with (III.1), we observe that \( c_{sk} = 0, \ 1 \leq s \leq 2, \ 3 \leq k \leq n \).

**ÖZET**

\( E^n \), n-boyutlu Öklid uzayında tamh 2-boyutlu regle yüzeylerinin minimal ve açılabılır olması için gerek ve yeter şartın total geodezik olması gösterildi ve \( M \) ile gösterilen bu yüzeyler için yen karakteristik özellikler bulundu. Ayrıca, 3-boyutlu Öklid uzayında tamh regle yüzeyler için iyi bilinen Massey teoremini, [3], \( E^n \) deki 2-regle yüzeyler için genelleştirildi.

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