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SOLIDITY AND SOME SEQUENCE SPACES

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SOLIDITY AND SOME SEQUENCE SPACES

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ABSTRACT:

In this paper we investigate the solidity (normality) of the sequence spaces $C_A, l_A, m_A$ and $\Gamma_A$.

1. INTRODUCTION AND NOTATION

We require the following sequence spaces:

c: the space of all convergent sequences.

m: the space of all bounded sequences.

l: the space of all sequences $x = \{x_k\}$ such that

$$\sum_{k=1}^{\infty} |x_k|$$

converges.

\(\Gamma\): the space of all sequences $x = \{x_k\}$ such that

$$|x_k|^{1/k} \to 0 \text{ as } k \to \infty$$

\(\omega\): the space of all sequences.

Let $A = (a_{nk})$, $(n, k = 1, 2, \ldots)$ be an infinite matrix. Given a sequence $x = \{x_k\}$ we write formally

$$y_n = A_n(x) = \sum_{k=1}^{\infty} a_{nk} \cdot x_k, \ (n = 1, 2, \ldots)$$

The sequence $\{y_n\} = \{A_n(x)\}$ will be denoted by $Ax$ or $y$. Let $X$ be a sequence space and let $X_A$ be the set of all those sequences $x = \{x_k\}$ for which $Ax \in X$. 
The set of all matrices transforming $X$ into $X$ will be denoted by $(X, X)$. We recall the following:

A sequence space $X$ is called solid (or normal) if an only if

$$m \, X \subset X.$$ 

Any matrix in $(c, c)$ is called a conservative matrix.

A conservative matrix which preserves the limit is said to be a Toeplitz-matrix.

2. RESULTS

PROPOSITION 1. If $A$ is a conservative matrix, which fails to sum a bounded sequence, then $c_A$ is not solid.

Proof.

The constant sequence $e = \{1, 1, \ldots \}$ is in $c_A$.

By our hypothesis, there exists a bounded sequence $b$ such that $b \notin c_A$.

That is $b \cdot e \notin c_A$ Therefore, $m \cdot c_A \notin c_A$ showing that $c_A$ is not solid.

COROLLARY. If $A$ is a Toeplitz matrix, then $c_A$ is not solid.

PROPOSITION 2. If $A \in (l, l)$, then $l_A$ is in general, not solid.

Proof.

Let

$$A = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 1 & \cdots \\
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
\end{pmatrix}$$

That is $a_{n, 2n-1} = 1$, ($n = 1, 2, \ldots$)

$a_{n, 2n} = 1$, ($n = 1, 2, \ldots$)

$a_{n,k} = 0$, otherwise.

Then

$$\sum_{n=1}^{\infty} |a_{nk}| = 1$$

for each fixed $k$, showing that $A \in (l, l)$
We note that
\[ x \in l_\Lambda \text{ if and only if } \sum_{k=1}^{\infty} |x_{2k-1} + x_{2k}| \text{ converges.} \]

Take \( x = \{1, -1, 1, -1, \ldots\} \) so that \( x \in l_\Lambda \).

Take \( b = \{1, -1, 1, -1, \ldots\} = x \). Then \( b \) is in \( m \).

Now, \( y = b x = \{1, 1, 1, \ldots\} = c \).

For \( c \), we have \( \sum_{k=1}^{\infty} |y_{2k-1} + y_{2k}| = 2 + 2 + \ldots \)

which is a divergent series.

Thus \( m \cdot l_\Lambda \subset l_\Lambda \). Hence, \( l_\Lambda \) is not solid.

PROPOSITION 3. If \( \Lambda \in (m, m) \), then \( m_\Lambda \) is in general, not solid.

Proof.

Let \( \Lambda = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -1 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} \)

In other words,

\[ \Lambda = (a_{nk}) \text{ is defined by} \]

\[ a_{n, 2n-1} = -1, \ (n = 1, 2, \ldots) \]

\[ a_{n, 2n} = 1, \ (n = 1, 2, \ldots) \]

\[ a_{nk} = 0, \text{ otherwise.} \]

Then

\[ \sum_{k=1}^{\infty} |a_{nk}| = 2 \text{ for each fixed } n. \]

Consequently, \( \Lambda \in (m, m) \).

Note that \( x \in m_\Lambda \) if and only if

\[ \Lambda x = \{-x_1 + x_2, -x_3 + x_4, -x_5 + x_6, \ldots\} \in m \]

We take \( x = \{1, 2, 3, \ldots\} \) so that
\( \Lambda x = \{1, 1, 1, \ldots \} \) and \( x \in \mathfrak{m}_\Lambda \).

Take \( b = \{-1, 1, -1, 1, \ldots \} \) in \( \mathfrak{m} \).

Then \( bx = \{-1, 1, -3, 4, \ldots \} \) and

\[ \Lambda (bx) = \{5, 7, 11, \ldots, 4n - 1, \ldots \} \notin \mathfrak{m}. \]

Thus \( \mathfrak{m} \mathfrak{m}_\Lambda \neq \mathfrak{m}_\Lambda \).

Hence \( \mathfrak{m}_\Lambda \) is not a solid space.

It is known [1] that a matrix \( A = (a_{nk}) \) is in \( (\Gamma, \Gamma) \) if and only if for each positive integer \( q \) there exists \( p_0 \geq q \) and a constant \( M(p, q) \) such that

\[
\sum_{n=0}^{\infty} \frac{|a_{nk}| q^n}{p^k} < M(p, q) \tag{1}
\]

for \( k = 0, 1, 2, \ldots \).

Here we take \( A = (a_{nk}), (n, k = 0, 1, 2, \ldots) \).

The above characterisation of the class \( (\Gamma, \Gamma) \) is equivalent to the following assertion, which we state as a Lemma.

**Lemma.** Let \( A = (a_{nk}), (n, k = 0, 1, 2, \ldots) \) be an infinite matrix. In order that the matrix \( A \) is in \( (\Gamma, \Gamma) \) it is necessary and sufficient that given any \( \varepsilon > 0 \), there is an \( M > 0 \), depending on \( \varepsilon \), such that uniformly in \( n \) and \( k \)

\[
a_{nk} = O\left(\varepsilon^n M^k\right) \tag{2}
\]

Proof:

Suppose that (1) holds. Since the terms in the sum on the left are all non-negative, each term is less than or equal to the sum, so that (1) implies that uniformly in \( n \) and \( k \)

\[
a_{nk} \cdot q^n = O\left(p^k\right) \tag{3}
\]

Given any \( \varepsilon > 0 \), choose an integer \( q \) with \( q > 1/\varepsilon \) and then choose \( p \) as in (1). Since \( q \geq 1/\varepsilon \) (3) gives us (2) with \( M = p \).

Conversely, suppose that (2) holds. Given any positive integer \( q \) choose \( \varepsilon < 1/q \), and then choose \( M \) as in (2).
Then
\[
\sum_{n=0}^{\infty} |a_{nk}| q^n = O \left\{ M^k \sum_{n=0}^{\infty} \varepsilon^n \cdot q^n \right\} \\
= O (M^k)
\]

Since \( \sum_{n=0}^{\infty} \varepsilon^n \cdot q^n \) converges (because \( \varepsilon < 1/q \)), it is equal to a constant. Thus if we take \( p \geq M \), then (1) holds. Hence the lemma.

**PROPOSITION 4.** If \( A \in (\Gamma, \Gamma) \), then \( \Gamma_A \) is not necessarily solid.

**Proof.**

Take \( A \) as in Proposition 3.

Writing \( \{t_m\} \) for the transform of \( \{x_n\} \), so that
\[
t_n = t_{2n-1}^m + x_{2n}, \ (n = 0, 1, 2, \ldots)
\]
we can verify directly that
\[
|x_n|^{1/n} \to 0 \Rightarrow |t_n|^{1/n} \to 0 \ (n \to \infty)
\]
For, if \( \eta < 1 \), then \( |a + b|^\eta < |a|^\eta + |b|^\eta \)
so that
\[
|t_n|^{1/n} < |x_{2n-1}|^{1/n} + |x_{2n}|^{1/n}
\]
(4)
since \( |x_n|^{1/n} \to 0 \ (n \to \infty) \), we have \( |x_n| < 1 \)
for sufficiently large \( n \). Supposing that \( n \) is large enough for
\[
|x_{2n-1}| < 1, \ |x_{2n}| < 1,
\]
\[
|t_n|^{1/n} < |x_{2n-1}|^{1/2n-1} + |x_{2n}|^{1/2n}
\]
Hence, if \( |x_n|^{1/n} \to 0 \), then \( |t_n|^{1/n} \to 0 \ (n \to \infty) \)
It is now trivial that \( \{1, 1, 1, \ldots\} \) belongs to \( \Gamma_A \) but \( \{-1, 1, -1, 1, \ldots\} \)
does not. Here we have \( b = \{-1, 1, -1, 1, \ldots\} \) in \( m \).
So, \( \Gamma_A \) is not solid.

3. **REMARK**

These results suggest some problems for consideration, which seem to be much harder. Take, for example, Proposition 2. Supposing that \( A \in (l,l) \), then \( l_A \) is not necessarily solid. But it may be solid. If \( l_A \)
= l (as happens, for example, when A is the identity transformation, though it will happen in some other cases as well), then \( I_A \) solid.

Again, suppose that \( I_A = \omega \).

It is easily seen that this will occur if and only if there is some \( k_0 \) such that

\[
a_{nk} = 0 \text{ for } k > k_0 \text{ and all } n; \tag{5}
\]

and for each fixed \( k \leq k_0 \), we have \( \{a_{nk}\} \in l \) \( \tag{6} \)

In this case also \( I_A \) is solid.

It seems that there is a plausible conjecture that these are the only cases. In other words we have the following conjecture.

**Conjecture 1.** Let \( A \in (l, l) \). Then \( I_A \) is solid if and only if either \( I_A = l \) or \( I_A = \omega \).

Analogous remarks apply to other propositions. There is, however one difference. In the case in which \( A \) gives the identity transformation, \( I_A = l \), which is solid. But \( c_A = c \), which is not solid. So, the corresponding conjecture in the case of Proposition 1 would be the following.

**Conjecture 2.** Let \( A \in (c, c) \). Then \( c_A \) is solid if and only if \( c_A = \omega \).

**Addendum.**

Take \( A = (a_{nk}) \), \( (n, k = 1, 2, 3, \ldots) \) as

\[
\begin{align*}
a_{nn} &= \frac{1}{n} \\
a_{nk} &= 0 \quad \text{if } k = n
\end{align*}
\]

\( (n, k = 1, 2, 3, \ldots) \).

Then certainly \( A \in (l, l) \). Also

\[
l_A = \{x = (x_k) : \sum_{k=1}^{\infty} \frac{|x_k|}{k} \text{ converges}\}.
\]

so that \( I_A \) is solid. However, \( I_A \neq l \)

(take \( x_k = \frac{1}{k} \) for \( k = 1, 2, 3, \ldots \)). and \( I_A \not\in w \) (take \( x_k = 1 \) for \( k = 1, 2, 3, \ldots \)) So the first conjecture false.

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