UMBRELLA MATRICES AND HIGHER CURVATURES OF A CURVE

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(Received: September 25, 1984)

ABSTRACT

In this paper, using the curvature matrix of a curve-hypersurface pair in the Cayley formula we obtain an umbrella matrix. Furthermore we give a relation between the Darboux matrix of the umbrella motion and the (higher) curvature matrix. In addition, using this umbrella matrix we also obtain an infinitesimal umbrella matrix.

BASIC CONCEPTS

Let M be a hypersurface in $E^n$ and $z$ be a curve, with the unit tangent vector field $Y_1$, which lies on M. Let the system $\{Y_1, \ldots, Y_{n-1}\}$, such that

$$ Y_i = \bar{D} Y_i Y_{i-1} , \quad 1 < i \leq n-1 $$

be linearly independent, where $\bar{D}$ is natural connection on M. Then the orthonormal system $\{X_1, \ldots, X_{n-1}\}$, which is obtained by Gram-Schmidt process from $\{Y_1, \ldots, Y_{n-1}\}$, is called the Frenet frame field of the curve $z$ in M. If we denote the unit normal vector field to M by $X_n$, then the orthonormal system $\{X_1, \ldots, X_{n-1}, X_n\}$ is called the natural frame field for the curve-hypersurface pair ($z$, M) [Hacisalihoğlu 1983].

DEFINITION 1: Let M be a hypersurface in $E^n$ and $z$ be a curve on M. Then the function

$$ k_{ig} : I \rightarrow \mathbb{R} $$

given by

$$ k_{ig} (s) = < X'_{i} (s) , X_{i+1} (s) > , $$
is called the $i^{th}$, $1 \leq i < n-1$, geodesic curvature function of the curve $\alpha$ and $k_{ig}(s)$ is called $i^{th}$ geodesic curvature of $\alpha$ at $s\in I$, where $I \subset \mathbb{R}$ [Guggenheimer (1963)].

**Theorem 1:** Let $M$ be a hypersurface in $E^n$ and $\alpha$ be a curve on $M$. The derivative formulas of the natural frame field $\{X_1, \ldots, X_{n-1}, X_n\}$ are

\[D_{X_i}X_i = X'_i = -k_{(i-1)g}X_{i-1} + k_{ig}X_{i+1} + \Pi(X_1, X_i)X_n,\]
\[D_{X_i}X_n = -\Pi(X_1, X_i)X_1 - \Pi(X_1, X_2)X_2 - \cdots - \Pi(X_1, X_{n-1})X_n\]

where $1 \leq i \leq n-1$ and $k_{og} = k_{(n-1)g} = 0$ [Guggenheimer (1963)].

In the matrix form, these derivative formulas become

\[
\begin{bmatrix}
X'_1 \\
X'_2 \\
\vdots \\
X'_{n-1} \\
X'_n
\end{bmatrix}
= \begin{bmatrix}
0 & k_{1g} & 0 & \cdots & 0 & 0 & \Pi(X_1, X_1) \\
-k_{1g} & 0 & k_{2g} & \cdots & 0 & 0 & \Pi(X_1, X_2) \\
& & & \ddots & & & \\
& & & & & \ddots & \\
0 & 0 & 0 & \cdots & -k_{(n-2)g} & 0 & \Pi(X_1, X_{n-1}) \\
-\Pi(X_1, X_1) & \cdots & \cdots & \cdots & -\Pi(X_1, X_{n-1}) & 0 & \Pi(X_1, X_n)
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_{n-1} \\
X_n
\end{bmatrix}
\]

or simply

\[X' = K(X)X.\]

The matrix $K(X)$ is called the (higher) curvature matrix of the pair $(\alpha, M)$ [Guggenheimer (1963)].

Let $y$ and $x$ be the position vectors, represented by column matrices, of a point $P$ in the fixed space $\Sigma^n$ and the moving space $E^n$, respectively. A continuous series of displacements, given by

\[y = Ax + b,\]

where the orthogonal matrix $A$ and the translation vector $b$ are functions of a parameter $s$ which may be identified with the time, is called a motion.
Now we consider the rotational motion, given by
\[ y = Ax. \]

The matrix
\[ W = A' A^t \]
is called the angular velocity matrix or the Darboux matrix of the motion \([\text{Bottema & Roth (1979)}]\).

**DEFINITION 2:** The orthogonal matrix A such that
\[ \text{AS} = S \]
is called an umbrella matrix, where \( S = [1\ 1\ \ldots\ 1]^t \in IR^n_1 \) \([\text{Özdamar (1977)}]\).

**DEFINITION 3:** Let A be an umbrella matrix. The motion generated by the transformation
\[
\begin{bmatrix}
y \\
1
\end{bmatrix} =
\begin{bmatrix}
A & C \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
1
\end{bmatrix}
\]
or
\[ y = Ax + C \]
is called an umbrella motion in \( E^n \), where \( x,y \in IR^n_1 \) and \( C \in IR^n_1 \) is the displacement vector of the origin \([\text{Haçsalihoglu (1977)}]\). Let \( B = [b_{ij}] \) be any nxn matrix and \( \varepsilon \) be an infinitesimal quantity of the first order. Then the matrix
\[ A = I_n + \varepsilon B \]
is called an infinitesimal matrix, where by \( I_n \), we denote the nxn unit matrix.

**UMBRELLA MATRICES AND HIGHER CURVATURES OF A CURVE**

In this section we assume that the directions \( X_2,X_3,\ldots,X_{n-2} \) of the natural frame field \( X = \{X_1,\ldots,X_n\} \) are conjugate directions with the tangent direction \( X_1 \) for a curve \( \alpha \) which is different from the line of curvature on a hypersurface \( M \) in \( E^n \). Then the higher curvature matrix can be written in the form
\[
\begin{bmatrix}
0 & k_{1g} & 0 & \ldots & 0 & \Pi(X_1,X_1) \\
-k_{1g} & 0 & k_{2g} & \ldots & 0 & 0 \\
0 & -k_{2g} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & \Pi(X_1,X_{n-1}) \\
-\Pi(X_1,X_1) & 0 & 0 & \ldots & -\Pi(X_1,X_{n-1}) & 0 
\end{bmatrix}
\]  

(1)

since

\[
\Pi(X_1,X_2) = \Pi(X_1,X_3) = \ldots = \Pi(X_1,X_{n-2}) = 0 .
\]

Let us write

\[
k_{ig} = b_{i-2} \quad (1 \leq i \leq n-2) \\
\Pi(X_1,X_{n-1}) = b_1 \\
-\Pi(X_1,X_1) = b_2 .
\]

Thus the equality (1), with respect to the elements \(b_j\) \((1 \leq j \leq n)\), takes the form

\[
\begin{bmatrix}
0 & b_3 & 0 & 0 & \ldots & 0 & 0 & -b_2 \\
-b_3 & 0 & b_4 & 0 & \ldots & 0 & 0 & 0 \\
0 & -b_4 & 0 & b_5 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & b_n & 0 \\
0 & 0 & 0 & 0 & \ldots & -b_n & 0 & b_1 \\
b_2 & 0 & 0 & 0 & \ldots & 0 & -b_1 & 0 
\end{bmatrix}
\]  

(2)
Let $A$ be any $n \times n$ orthogonal matrix one of the characteristic values of which is not $-1$. Then $A$ can always be expressed as

$$A = (I_n - B)^{-1} (I_n + B),$$

where $B$ is as $n \times n$ skew matrix. The formula (3) is known as the Cayley formula. Now taking $K(X)$ instead of $B$ in this formula we obtain the following theorem.

**THEOREM 1:** For $n \geq 3$, if $b_1 = b_2 = \ldots = b_n = c$ then the orthogonal matrix

$$A = (I_n - K(X))^{-1} (I_n + K(X))$$

one of the characteristic values of which is not $-1$, is an umbrella matrix.

**PROOF:** In order to prove the theorem it suffices to show the equality

$$A S = S,$$

where $S = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^t \in \mathbb{R}^n_1$.

$$(I_n + K(X)) S = S + K(X) S$$

$$= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & -1 \\ -1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \ldots & 0 & -1 & 0 \\ 1 & -1 & 0 & \ldots & 0 & -1 & 1 \end{bmatrix}$$

$$= S.$$
This means that $S$ is in the kernel of $K(X)$. Thus we have

$$ (I_n - K(X)) S = S $$

from which we get

$$ S = (I_n - K(X))^{-1} S. $$

Then

$$ AS = (I_n - K(X))^{-1} (I_n + K(X)) S $$

$$ = (I_n - K(X))^{-1} S $$

$$ = S, $$

which completes the proof.

Now we may give a relation between the Darboux matrix of the umbrella motion and the higher curvature matrix by the following theorem.

THEOREM 2: Let $W(A)$ be the Darboux matrix of the umbrella motion, where $A$ is given by (4), and $K(X)$ be the higher curvature matrix. Then

$$ W(A) = \frac{2c'}{c} (I_n - K(X))^{-1} K(X) (I_n + K(X))^{-1}, $$

(5)

where $c = c(s)$.

PROOF: Differentiating (II.4), with respect to $s$, we have

$$ A' \left[ (I_n - K(X))^{-1} \right]' (I_n + K(X)) + (I_n - K(X))^{-1} (I_n + K (X))' $$

$$ = - (I_n - K(X))^{-1} (I_n - K(X))' (I_n - K(X))^{-1} (I_n + K(X)) $$

$$ + (I_n - K(X))^{-1} K'(X) $$

$$ = (I_n - K(X))^{-1} K'(X) (I_n - K(X))^{-1} (I_n + K(X)) $$

$$ + (I_n - K(X))^{-1} K'(X) $$

$$ = (I_n - K(X))^{-1} K'(X) \left[ (I_n - K(X))^{-1} (I_n + K(X)) + I_n \right]. $$

In the last expression if we write

$$ I_n = (I_n - K(X))^{-1} (I_n - K(X)) $$

then we also have

$$ A' = 2 (I_n - K(X))^{-1} K'(X) (I_n - K(X))^{-1}. $$

From

$$ K(X) = cU \quad (U \text{ is constant matrix}), $$
\[ K'(X) = \frac{e'}{c} K(X) \]

it follows that

\[ A' = \frac{2e'}{c} (I_n - K(X))^{-1} K(X) (I_n - K(X))^{-1} \]

Thus from \( W(A) = A' A^t \) we obtain

\[
W(A) = \frac{2e'}{c} (I_n - K(X))^{-1} K(X) (I_n - K(X))^{-1} (I_n + K(X))^{-1} (I_n - K(X))^{-1} \\
= \frac{2e'}{c} (I_n - K(X))^{-1} K(X) [(I_n - K(X))^{-1} (I_n + K(X))^{-1} (I_n - K(X))]^{-1} (I_n - K(X)) \\
= \frac{2e'}{c} (I_n - K(X))^{-1} K(X) [(I_n - K(X))^{-1} (I_n + K(X))^{-1} (I_n - K(X))]^{-1} (I_n - K(X)) \\
= \frac{2e'}{c} (I_n - K(X))^{-1} K(X) (I_n + K(X))^{-1} (I_n - K(X))^{-1} (I_n - K(X)) \\
= \frac{2e'}{c} (I_n - K(X))^{-1} K(X) (I_n + K(X))^{-1} .
\]

**THEOREM 3:** Let \( W(A) \) be the Darboux matrix of the umbrella motion, where \( A \) is given by (4). Then the matrix \( I_n + W(A) \) ds is also an (infinitesimal) umbrella matrix.

**PROOF:** We can write

\[ Y + Y' ds = AX + C + (A'X + C') ds \]

for an infinitesimal motion in \( \mathbb{E}^n \). Thus since

\[ X = A^t (Y - C) \]

we have

\[ Y + Y' ds = (I_n + A'A^t ds) (Y - C) + C + C' ds \]

or by \( W(A) = A'A^t \)

\[ (Y' - C') ds = (I_n + W(A)ds) (Y - C) - (Y - C) . \]

Moreover since

\[ (I_n + W(A)ds) - I_n = W(A) \] ds
the matrix $I_n + W(A) \, ds$ is an infinitesimal orthogonal matrix. Thus for $S = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^t \in \mathbb{R}^n_1$ we get

$$(I_n + W(A)ds) \, S = S + W(A)ds \, S$$

$$= S + \frac{2c'}{c} \left( I_n - K(X) \right)^{-1} K(X) \left( I_n + K(X) \right)^{-1} S$$

$$= S + \frac{2c'}{c} \left( I_n - K(X) \right)^{-1} K(X) \, S.$$ 

In the last expression we can write

$$K(X) \, S = 0$$

since $S$ is in the kernel of $K(X)$, and so we obtain

$$(I_n + W(A)ds) \, S = S$$

which means that the matrix $I_n + W(A)ds$ is an (indifinitesimal) umbrella matrix.

REFERENCES

BOTTEMA, O., and ROTH, B., 1979. *Theoretical Kinematics*, North Holland,


