CONVOLUTIONS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS
WITH NEGATIVE COEFFICIENTS

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ABSTRACT

Let \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \), \( a_n \geq 0 \) and \( g(z) = z - \sum_{n=2}^{\infty} b_n z^n \), \( b_n \geq 0 \). We investigate some properties of \( h(z) = f(z) * g(z) \) where \( f(z) \) and \( g(z) \) satisfy either \( \text{Re}(f(z)/z) > \alpha \), \( \text{Re}(g(z)/z) > \alpha \) or \( \text{Re} f''(z) > \alpha \), \( \text{Re} g'(z) > \alpha \) for \( |z| < 1 \).

INTRODUCTION

Let \( S \) denote the class of functions normalized by \( f(0) = f'(0) - 1 = 0 \) that are analytic and univalent in the unit disk \( E \). A function \( f(z) \in S \) is said to be starlike if \( \text{Re} \left( z f'(z) / f(z) \right) > 0 \) for \( |z| < 1 \) and is said to be convex if \( \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0 \) for \( |z| < 1 \). These classes are denoted by \( S^* \) and \( K \) respectively.

The convolution or Hadamard product of two power series

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n
\]

is defined as the power series \( (f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \).

Ruscheweyh and T. Sheil–small (1973) proved the Polya–Schoenberg conjecture that if \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K \) and
\[ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in K, \]

then
\[ h(z) = f(z) \ast g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \in K. \]

Let \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \), \( a_n \geq 0 \) and let \( P(\alpha) \) denote the class of functions of the form \( f'(z) \) which satisfy \( \text{Re}(f(z)/z) > \alpha \) for \( |z| < 1 \) and \( Q(\alpha) \) denote the class of functions \( f(z) \) which satisfy \( \text{Re} f'(z) > \alpha \) for \( |z| < 1 \). In this paper we obtain some properties of \( h(z) = f(z) \ast g(z) \) where \( f(z) \) and \( g(z) \) belong to \( P(\alpha) \) or \( Q(\alpha) \) for \( 0 \leq \alpha < 1 \).

Schild and Silverman (1975) investigated some properties of convolutions of univalent functions with negative coefficients.

**CONVOLUTION PROPERTIES**

We need the following result:

**LEMMA:**

i. \( f(z) \in P(\alpha) \iff \sum_{n=2}^{\infty} a_n \leq 1 - \alpha \)

ii. \( f(z) \in Q(\alpha) \iff \sum_{n=2}^{\infty} n a_n \leq 1 - \alpha \).

**PROOF:** The lemma has been proved in Sarangi and Uralegaddi (1978)

**THEOREM 1:** If \( f(z) \in P(\alpha) \) and \( g(z) \in P(\alpha) \) then \( h(z) = f(z) \ast g(z) \)
\[ = z - \sum_{n=2}^{\infty} a_n b_n z^n \in P(2\alpha - \alpha^2). \]

**PROOF:** From Lemma we have
\[ \sum_{n=2}^{\infty} a_n \leq 1 - \alpha \text{ and } \sum_{n=2}^{\infty} b_n \leq 1 - \alpha . \]

In view of Lemma, we have to find the largest \( \beta = \beta(\alpha) \) such that
\[ \sum_{n=2}^{\infty} a_n b_n \leq 1 - \beta. \]

We have to show that

\[ \sum_{n=2}^{\infty} \frac{a_n}{1 - \alpha} \leq 1 \quad (1) \]

and

\[ \sum_{n=2}^{\infty} \frac{b_n}{1 - \alpha} \leq 1 \quad (2) \]

imply that

\[ \sum_{n=2}^{\infty} \frac{a_n b_n}{1 - \beta} \leq 1 \text{ for all } \beta = \beta (\alpha) = 2 \alpha - \alpha^2. \quad (3) \]

From (1) and (2) we obtain by means of Cauchy–Schwarz inequality

\[ \sum_{n=2}^{\infty} \frac{\sqrt{a_n} \sqrt{b_n}}{1 - \alpha} \leq 1 \quad (4) \]

Hence it is sufficient to prove that

\[ \frac{a_n b_n}{1 - \beta} \leq \frac{\sqrt{a_n} \sqrt{b_n}}{1 - \alpha}, \beta = \beta (\alpha), n = 2, 3 \ldots \text{ or } \sqrt{a_n} \sqrt{b_n} \leq \frac{1 - \beta}{1 - \alpha} \]

From (4) we have \[ \sqrt{a_n} \sqrt{b_n} \leq 1 - \alpha \text{ for each } n. \] Hence it will be sufficient to show that

\[ 1 - \alpha \leq \frac{1 - \beta}{1 - \alpha} \quad (5) \]

Solving for \( \beta \) we get \( \beta \leq 2 \alpha - \alpha^2. \)

The result is sharp with equality for \( f(z) = g(z) = z - (1 - \alpha) z^2. \)

COROLLARY: Let \( f(z) \in P(\alpha) \), \( g(z) \in P(\alpha) \) and let

\[ h(z) = z - \sum_{n=2}^{\infty} \sqrt{a_n} \sqrt{b_n} z^n. \] Then \( \Re (h(z)/z) > \alpha \) for \( |z| < 1. \)
This result follows from the inequality (4). It is sharp for the same
functions as in Theorem 1.

THEOREM 2. Let \( f(z) \in P(\alpha) \) and \( g(z) \in P(\beta) \), then
\[
h(z) = f(z) \ast g(z) \in P(\alpha + \beta - \alpha \beta).
\]

PROOF: The proof is similar to that of Theorem 1.

COROLLARY: Let \( f(z) \in P(\alpha) \), \( g(z) \in P(\beta) \) and \( h(z) \in P(\Gamma) \), then
\[
f(z) \ast g(z) \ast h(z) \in P(\alpha + \beta + \Gamma - \alpha \beta - \beta \Gamma - \Gamma \alpha + \alpha \beta \Gamma).
\]

THEOREM 3: Let \( f(z) \in Q(\alpha) \) and \( g(z) \in Q(\beta) \), then
\[
h(z) = f(z) \ast g(z) \in Q\left(\frac{1 + \alpha + \beta - \alpha \beta}{2}\right).
\]

PROOF: From Lemma, we know that
\[
\sum_{n=2}^{\infty} \frac{n a_n}{1 - \alpha} \leq 1 \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{n b_n}{1 - \beta} \leq 1.
\]
We have to find the largest \( \Gamma = \Gamma (\alpha, \beta) \) such that
\[
\sum_{n=2}^{\infty} n a_n b_n \leq 1 - \Gamma.
\]

It is sufficient to show that \( \sum_{n=2}^{\infty} \frac{n a_n}{1 - \alpha} \leq 1 \) and \( \sum_{n=2}^{\infty} \frac{n b_n}{1 - \beta} \leq 1 \)

imply \( \sum_{n=2}^{\infty} \frac{n a_n b_n}{1 - \Gamma} \leq 1 \) for all \( \Gamma = \Gamma (\alpha, \beta) = \frac{1 + \alpha + \beta - \alpha \beta}{2} \).

Proceeding similarly as in the proof of Theorem 1 we get
\[
\frac{a_n b_n}{1 - \Gamma} \leq \frac{n a_n b_n}{(1 - \alpha)(1 - \beta)} \quad \text{or} \quad \Gamma \leq 1 - \frac{(1 - \alpha)(1 - \beta)}{n}
\]
The right-hand side is an increasing function of \( n \) (\( n=2, 3, \ldots \)). Taking
\( n=2 \), we get \( \Gamma \leq \frac{1 + \alpha + \beta - \alpha \beta}{2} \).
THEOREM 4: Let \( f(z) \in Q(\alpha) \) and \( g(z) \in Q(\beta) \). Then
\[
f(z) \ast g(z) \in P \left( \frac{3 + \alpha + \beta - \alpha \beta}{4} \right).
\]

PROOF: From Lemma, we have
\[
\sum_{n=2}^{\infty} n a_n \leq 1 - \alpha \quad \text{and} \quad \sum_{n=2}^{\infty} n b_n \leq 1 - \beta.
\]
We have to find the largest \( \Gamma = \Gamma(\alpha, \beta) \) such that
\[
\sum_{n=2}^{\infty} a_n b_n \leq 1 - \Gamma.
\]
This is satisfied if
\[
\frac{1}{1-\Gamma} \leq \frac{n^2}{(1-\alpha)(1-\beta)} \quad \text{i.e., for} \quad \Gamma \leq 1 - \frac{(1-\alpha)(1-\beta)}{n^2},
\]
Since the right-hand side is an increasing function of \( n \), taking \( n = 2 \)
we get the result.

THEOREM 5: If \( f, g \in Q(\alpha) \), then
\[
h(z) = z - \sum_{n=2}^{\infty} (a_n^2 + b_n^2) z^n \in Q(2z - \alpha^2).
\]

PROOF: Since \( \sum_{n=2}^{\infty} n a_n \leq 1 - \alpha \), we have
\[
\sum_{n=2}^{\infty} \frac{n^2 a_n^2}{(1-\alpha)^2} \leq \left( \sum_{n=2}^{\infty} \frac{n a_n}{1-\alpha} \right)^2 \leq 1.
\]
Similarly,
\[
\sum_{n=2}^{\infty} \frac{n^2 b_n^2}{(1-\alpha)^2} \leq 1 \quad \text{and therefore}
\]
\[
\sum_{n=2}^{\infty} \frac{1}{2} n^2 \frac{a_n^2 + b_n^2}{(1-\alpha)^2} \leq 1.
\]
We have to find the largest \( \beta = \beta(\alpha) \) such that
\[
\sum_{n=2}^{\infty} \frac{n}{1-\beta} (a_n^2 + b_n^2) \leq 1.
\]
This will be satisfied if
\[ \frac{1}{1 - \beta} \leq \frac{1}{2} \frac{n}{(1 - \alpha)^2} \]

or \( \beta \leq 1 - \frac{2}{n} (1 - \alpha)^2 \).

Again since the right-hand side is an increasing function of \( n \), we get
\[ \beta \leq 2 (1 - \alpha)^2. \]

NOTE: The result is sharp for the functions
\[ f(z) = g(z) = z - \frac{1}{2} (1 - \alpha) z^2. \]

REFERENCES

