DECOMPOSITIONS OF SOME FORMS OF CONTINUITY

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Abstract. In this paper, $\alpha_I N_3$–sets [3] and $\alpha_I N_5$–sets [3] are introduced and characterizations of $\alpha - I$–open [6], semi $- I$–open [6], $\alpha_I N_3$– and $\alpha_I N_5$–sets are investigated. Also, new decompositions of $\alpha - I$–continuity and semi $- I$–continuity are obtained using these sets.

1. Introduction

Quite recently, Acikgoz and Yuksel [3] have introduced $I$ - R closed sets and obtained a decomposition of continuity. Acikgoz et al. [1], [2] investigated some properties of $I$ - open sets and obtained decompositions of $I$ - continuity and semi $- I$ - continuity.

The purpose of this paper is to introduce $\alpha_I N_3$ – sets and $\alpha_I N_5$ – sets via idealization and investigate characterizations of $\alpha - I$ – open, semi $- I$ – open, $\alpha_I N_3$ – and $\alpha_I N_5$ – sets and also, to obtain new decompositions of $\alpha - I$ – continuity and semi $- I$ – continuity using these sets.

2. Preliminaries

Throughout this paper $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure and the interior of $A$, respectively. Let $(X, \tau)$ be a topological space and let $I$ be an ideal of subsets of $X$. An ideal is defined as a nonempty collection $I$ of subsets of $X$ satisfying the following two conditions: (1) If $A \in I$ and $B \subseteq A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$. An ideal topological space is a topological space $(X, \tau)$ with an ideal $I$ on $X$ and is denoted by $(X, \tau, I)$. For a subset $A \subseteq X$, $A^\ast (I) = \{ x \in X : U \cap A \notin I$ for each neighborhood $U$ of $x \}$ is called the local function of $A$ with respect to $I$ and $\tau$ [10]. We simply write $A^\ast$ instead of $A^\ast (I)$ since there is no chance for confusion. $X^\ast$ is often a proper subset of $X$. The hypothesis

$X = X^\ast$ [9] is equivalent to the hypothesis $\tau \cap I = \emptyset$ [13]. The ideal topological space which satisfies this hypothesis is called a Hayashi – Samuels space [10]. For

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every ideal topological space \((X, \tau, I)\), there exists a topology \(\tau^*(I)\), finer than \(\tau\), generated by \(\beta(I, \tau) = \{U \setminus I : U \in \tau \text{ and } I \subseteq U\}\), but in general \(\beta(I, \tau)\) is not always a topology [10]. Additionally, \(\text{Cl}^*(A) = A \cup A^*\) defines a Kuratowski closure operator for \(\tau^*(I)\).

First we shall recall some definitions used in the sequel.

**Definition 2.1.** A subset \(A\) of an ideal topological space \((X, \tau, I)\) is said to be

1. \(\alpha - I - \text{open} [6]\) if \(A \subseteq \text{Int}(\text{Cl}^*(\text{Int}(A)))\),
2. \(\text{pre} - I - \text{open} [4]\) if \(A \subseteq \text{Int}(\text{Cl}^*(A))\),
3. \(\text{semi} - I - \text{open} [6]\) if \(A \subseteq \text{Cl}^*(\text{Int}(A))\),
4. \(\delta - I - \text{open} [2]\) if \(\text{Int}(\text{Cl}^*(A)) \subseteq \text{Cl}^*(\text{Int}(A))\),
5. \(\text{strong} \beta - I - \text{open} [8]\) if \(A \subseteq \text{Cl}^*(\text{Int}(\text{Cl}^*(A)))\),
6. \(t - I - \text{set} [6]\) if \(\text{Int}(A) = \text{Int}(\text{Cl}^*(A))\),
7. \(\tau^* - \text{dense set} [9]\) if \(X = \text{Cl}^*(A)\),
8. \(\tau^* - \text{closed set} [10]\) if \(A = \text{Cl}^*(A)\).

The family of all \(\alpha - I - \text{open}\) (resp. \(\text{pre} - I - \text{open}, \text{semi} - I - \text{open}, \text{strong} \beta - I - \text{open}\) ) sets in an ideal topological space \((X, \tau, I)\) is denoted by \(\alpha IO(X, \tau)\) (resp. \(\Pi O(X, \tau), \Sigma IO(X, \tau)\)).

**Definition 2.2.** A subset \(A\) of an ideal topological space \((X, \tau, I)\) is said to be

1. \(a \text{ semi} - I - \text{closed} [7]\) if \(\text{Int}(\text{Cl}^*(A)) \subseteq A\),
2. \(a \text{ weakly} I - \text{locally} - \text{closed set} [11]\) if \(A = U \cap V\), where \(U\) is open and \(V\) is \(\tau^* - \text{closed}\),
3. \(a B_I - \text{set} [6]\) if \(A = U \cap V\), where \(U\) is open and \(V\) is a \(t - I - \text{set}\).

The family of all \(a \text{ semi} - I - \text{closed}\) (resp. \(a \text{ weakly} I - \text{locally} - \text{closed}, B_I - \) ) sets in an ideal topological space \((X, \tau, I)\) is denoted by \(\Sigma IC(X, \tau)\) (resp. \(W_I LC(X, \tau), B_I(X, \tau)\)).

**Definition 2.3.** A subset \(A\) of an ideal topological space \((X, \tau, I)\) is said to be a

\(\text{nowhere} \ \tau^* - \text{dense set}\) if \(\text{Int}^*(\text{Cl}(A)) = \emptyset\), where \(\text{Int}^*(A)\) denotes the interior of \(A\) with respect to \(\tau^*\).

**Theorem 2.4.** A subset \(A\) of a space \((X, \tau, I)\) is semi - I - closed if and only if \(\text{Int}(\text{Cl}^*(A)) = \text{Int}(A)\).

**Proof.** Necessity. Let \(A\) be semi - I - closed. Then we have \(\text{Int}(\text{Cl}^*(A)) \subseteq A\). Then \(\text{Int}(\text{Cl}^*(A)) \subseteq \text{Int}(A)\) and hence \(\text{Int}(\text{Cl}^*(A)) \subseteq \text{Int}(A)\).

The sufficiency is clear. \(\square\)

**Corollary 1.** Let \(A\) be a subset of an ideal topological space \((X, \tau, I)\). \(A\) is a semi - I - closed set if and only if \(A = A \cup \text{Int}(\text{Cl}^*(A))\).

**Proof.** The necessity is clear as seen in Theorem 1.
Sufficiency.

\[
\text{Int}(\text{Cl}^*(B)) = \text{Int}(\text{Cl}^*(A \cup \text{Int}(\text{Cl}^*(A))))
\]

\[
\subseteq \text{Int}(\text{Cl}^*(A) \cup \text{Int}(\text{Cl}^*(\text{Int}(\text{Cl}^*(A))))
\]

\[
= \text{Int}(\text{Cl}^*(A)) \subseteq A \cup \text{Int}(\text{Cl}^*(A)) = B.
\]

Thus we obtain that \(\text{Int}(\text{Cl}^*(B)) \subseteq B\) and hence \(B = A \cup \text{Int}(\text{Cl}^*(A))\) is semi-I-closed. \(\square\)

**Definition 2.5.** A subset \(A\) of a space \((X, \tau, I)\) is said to be \(\beta-I\)-closed if its complement is \(\beta-I\)-open.

The family of all \(\beta-I\)-closed sets in an ideal topological space \((X, \tau, I)\) is denoted by \(\beta \text{IC} (X, \tau)\).

**Theorem 2.6.** Let \(A\) be subset of an ideal topological space \((X, \tau, I)\). Then, If \(A\) is \(\beta-I\)-closed, then \(\text{Int}(\text{Cl}^*(\text{Int}(A))) \subseteq A\).

**Proof.** Since \(A\) is \(\beta-I\)-closed, \(X - A \in \beta \text{IO} (X, \tau)\). Since \(\tau^*(I)\) is finer than \(\tau\), we have

\[
X - A \subseteq \text{Cl}(\text{Int}(\text{Cl}^*(X - A)))
\]

\[
\subseteq \text{Cl}(\text{Int}(\text{Cl}(X - A)))
\]

\[
= X - \text{Int}(\text{Cl}(\text{Int}(A)))
\]

\[
\subseteq X - \text{Int}(\text{Cl}^*(\text{Int}(A))).
\]

Therefore, we obtain \(\text{Int}(\text{Cl}^*(\text{Int}(A))) \subseteq A\). \(\square\)

3. \(\alpha I N_3\) sets

**Proposition 1.** Every semi-I-closed set of an ideal topological space is \(\beta-I\)-closed.

**Proof.** Let \(A\) be semi-I-closed. Then we have \(\text{Int}(\text{Cl}^*(A)) \subseteq A\). Then \(\text{Int}(\text{Cl}^*(\text{Int}(A))) \subseteq \text{Int}(\text{Cl}^*(A)) \subseteq \text{Int}(A)\) and hence \(\text{Int}(\text{Cl}^*(\text{Int}(A))) \subseteq A\). \(\square\)

**Remark 3.1.** The converses of Proposition 1 need not be true as shown in the following example.

**Example 3.2.** Let \(X = \{1, 2, 3, 4\}, \tau = \{\emptyset, X, \{1, 2\}, \{4\}, \{1, 2, 4\}\}\) and \(I = \{\emptyset, \{3\}\}\). Then \(A = \{2, 3, 4\}\) is a \(\beta-I\)-closed set, but not semi-I-closed. For, \(\text{Int}(\text{Cl}^*(\text{Int}(A))) = \text{Int}(\text{Cl}^*(\{2, 3, 4\})) = \text{Int}(\text{Cl}^*(\{4\})) = \text{Int}(\{4\}) \subseteq \{4\} \subseteq \text{Int}({3, 4} \cup \{4\}) = \text{Int}({3, 4}) = \{4\}\) and hence \(\text{Int}(\text{Cl}^*(\text{Int}(\{2, 3, 4\}))) = \{4\} \subseteq \{2, 3, 4\} = A\). This shows that \(A\) is a \(\beta-I\)-closed set. But \(A\) is not a semi-I-closed set. For \(\text{Int}(\text{Cl}^*(A)) = \text{Int}(\text{Cl}^*(\{2, 3, 4\})) = \text{Int}({\{2, 3, 4\}} \cup \{2, 3, 4\}) = \text{Int}(X \cup \{2, 3, 4\}) = \text{Int}(X) = X \not\subseteq \{2, 3, 4\} = A\) and \(\text{Int}(\text{Cl}^*(A)) \not\subseteq A\).

**Lemma 3.3.** (Acikgoz et. al [1]). Let \((X, \tau, I)\) be an ideal topological space and \(A\) a subset of \(X\). Then the following properties hold:
(1) If \( O \) is open in \((X, \tau, I)\), then \( O \cap \text{Cl}^*(A) \subset \text{Cl}^*(O \cap A) \).

(2) If \( A \subset X_0 \subset X \), then \( \text{Cl}^*(A) \subset \text{Cl}^*(X_0 \cap A) \).

**Proposition 2.** Let \((X, \tau, I)\) be an ideal topological space. \( A \in \alpha IO(X, \tau) \) if and only if \( A \cap S \in \text{SISO} (X, \tau) \) for each \( S \in \text{SISO} (X, \tau) \).

**Proof.** Necessity. Let \( A \in \alpha IO(X, \tau) \) and \( S \in \text{SISO} (X, \tau) \). Using Lemma 1, we obtain

\[
S \cap A \subset \text{Cl}^*(\text{Int}(S)) \cap \text{Int}^*(\text{Int}(\text{Cl}^*(A)))
\]

As \( \text{Cl}^*(\text{Int}(A)) \subset \text{Cl}^*(S) \), we have

\[
S \cap A \subset \text{Cl}^*(\text{Int}(S)) \cap \text{Int}^*(\text{Int}(\text{Cl}^*(A)))
\]

This shows that \( A \cap S \in \text{SISO} (X, \tau) \).

Sufficiency. Let \( S \in \text{SISO} (X, \tau) \) and \( A \cap S \in \text{SISO} (X, \tau) \). Then in particular \( A \in \text{SISO} (X, \tau) \). Assume \( x \in A \cap \text{C}(\text{Int}(\text{Cl}^*(A))) \) (\( \text{C} \) denoting complement). Then \( x \in \text{Cl}^*(S) = \text{Cl}^*(\text{Int}(S)) \) by [7], where \( S = \text{C}(\text{Cl}^*(\text{Int}(A))) \). Hence we obtain

\[
S \cup \text{Int}\{x\} \subset \text{Cl}^*(\text{Int}(S))
\]

Thus \( S \cup \{x\} \in \text{SISO} (X, \tau) \) and consequently \( A \cap (S \cup \{x\}) \in \text{SISO} (X, \tau) \). But

\[
A \cap (S \cup \{x\}) = (A \cap S) \cup (A \cap \{x\})
\]

Hence \( \{x\} \) is open. As \( x \in \text{Cl}^*(\text{Int}(A)) \), this implies \( x \in \text{Int}(\text{Cl}^*(\text{Int}(A))) \), contrary to assumption. Thus \( x \in A \) implies \( x \in \text{Int}(\text{Cl}^*(\text{Int}(A))) \), and \( A \in \alpha IO (X, \tau) \). This completes the proof. \( \square \)

**Proposition 3.** Let \((X, \tau, I)\) be an ideal topological space. \( A \in \alpha IO (X, \tau) \) if and only if \( A = U \cap D \) where \( U \in \tau \) and \( \text{Int}(D) \) is \( \tau^* - \text{dense} \).

**Proof.** Necessity. If \( A \in \alpha IO (X, \tau) \), then we have

\[
A = \text{Int}(\text{Cl}^*(\text{Int}(A))) - (\text{Int}(\text{Cl}^*(\text{Int}(A))) - A)
\]

where \( \text{Int}(\text{Cl}^*(\text{Int}(A))) = U \in \tau \) and \( \text{Int}(\text{Cl}^*(\text{Int}(A))) - A \) is nowhere \( \tau^* - \text{dense} \).

Sufficiency. Let \( A = U \cap D \), where \( U \in \tau \) and \( \text{Int}(D) \) is \( \tau^* - \text{dense} \). Since \( A \subset U \)
\[ U = U \cap X = U \cap \text{Cl}^*(\text{Int}(D)) \subseteq \text{Int}(U) \cap \text{Cl}^*(\text{Int}(D)) \subseteq \text{Cl}^*(\text{Int}(U) \cap \text{Int}(D)) = \text{Cl}^*(\text{Int}(A)) \] and we obtain \( U \subseteq \text{Int}(\text{Cl}^*(\text{Int}(A))) \). Hence \( A \subseteq \text{Int}(\text{Cl}^*(\text{Int}(A))) \) so that \( A \in \alpha \text{IO} (X, \tau) \).

**Definition 3.4.** [3]. A subset \( H \) of an ideal topological space \( (X, \tau, I) \) is called an \( \alpha_I N_3 \)-set if \( H = A \setminus B \) where \( A \in \alpha \text{IO} (X, \tau) \) and \( B \) is a \( t - I \)-set.

The family of all \( \alpha_I N_3 \)-sets of \( (X, \tau, I) \) is denoted by \( \alpha_I N_3 (X, \tau) \) in this paper, when there is no chance for confusion with the ideal.

**Theorem 3.5.** For a subset \( A \) of an ideal topological space \( (X, \tau, I) \), the following properties are equivalent:

1. \( A \) is semi-\( I \)-closed,
2. \( A \) is \( \beta - I \)-closed and is an \( \alpha_I N_3 \)-set,
3. \( A \) is \( \beta - I \)-closed and \( \delta - I \)-open.

**Proof.** a) \( \Rightarrow \) b). Let \( A \in \text{SIC}(X, \tau) \). Since \( \text{SIC}(X, \tau) \subseteq \text{\beta IC}(X, \tau) \) by Proposition 1 and \( A = A \cap X \), where \( A \) is a \( t - I \)-set and \( X \in \alpha \text{IO} (X, \tau) \). Therefore we have \( \text{SIC}(X, \tau) \subseteq \beta \text{IC}(X, \tau) \cap \alpha_I N_3 (X, \tau) \).

b) \( \Rightarrow \) c). The proof is seen in the Diagram.

c) \( \Rightarrow \) a). Let \( A \) be \( \beta - I \)-closed and \( \delta - I \)-open. Since \( \text{Int}(\text{Cl}^*(A)) \subseteq \text{Int}(\text{Cl}^*(\text{Int}(A))) \) and \( \text{Int}(\text{Cl}^*(\text{Int}(A))) \subseteq A \), we obtain that \( A \in \text{SIC}(X, \tau) \).

**Proposition 4.** Let \( (X, \tau, I) \) be an ideal topological space. \( H \in \alpha_I N_3 (X, \tau) \) if and only if \( H = B \cap D \) where \( B \in B_I (X, \tau) \) and \( \text{Int}(D) \) is \( \tau^* \)-dense.

**Proof.** Necessity. Let \( H \in \alpha_I N_3 (X, \tau) \) and write \( H = A \cap B \) where \( A \in \alpha \text{IO} (X, \tau) \) and \( B \) is a \( t - I \)-set. By Proposition 3, we write \( A = U \cap D \) where \( U \in \tau \) and \( \text{Int}(D) \) is \( \tau^* \)-dense. Thus \( H = A \cap B = (U \cap D) \cap B = (U \cap B) \cap D \) where \( U \cap B \in B_I (X, \tau) \) and \( \text{Int}(D) \) is \( \tau^* \)-dense as required.

Sufficiency. Assume that \( H = B \cap D \) with \( B \in B_I (X, \tau) \) and \( \text{Int}(D) \) is \( \tau^* \)-dense. Then we have \( B = U \cap B_2 \) where \( U \in \tau \) and \( B_2 \) is a \( t - I \)-set. Thus \( H = B \cap D = (U \cap B_2) \cap D = (U \cap D) \cap B_2 \) where \( U \cap D \in \alpha \text{IO} (X, \tau) \) by Proposition 3 and \( B_2 \) is a \( t - I \)-set. Therefore we obtain \( H \in \alpha_I N_3 (X, \tau) \).

**Proposition 5.** Let \( (X, \tau, I) \) be an ideal topological space. \( H \in \alpha_I N_3 (X, \tau) \) if and only if \( H = A \cap (H \cup \text{Int}(\text{Cl}^*(H))) \) where \( A \in \alpha \text{IO} (X, \tau) \).
Proof. Necessity. Let \( H \in \alpha_I N_3 (X,\tau) \) and assume \( H = A \cap B \) where \( A \in \alpha IO (X,\tau) \) and \( B \) is a \( t-I \) set. Since \( B \) is a \( t-I \) set, we have

\[
H = A \cap (H \cup Int(Cl^*(H))) = A \cap (B \cup Int(Cl^*(B))) = A \cap B = H.
\]

So \( H = A \cap (H \cup Int(Cl^*(H))) \), with \( A \in \alpha IO (X,\tau) \) by Lemma 1 as required.

Sufficiency. Assume that \( H \subseteq X \) such that \( H = A \cap (H \cup Int(Cl^*(H))) \) where \( A \in \alpha IO (X,\tau) \).

Since \( H \cup Int(Cl^*(H)) \) is semi \( -I \) closed by Corollary 1 and hence it is a \( t-I \) set. Therefore \( H \in \alpha_I N_3 (X,\tau) \).

**Theorem 3.6.** Let \((X, \tau, I)\) be an ideal topological space.

\[
\alpha IO (X,\tau) = PIO (X,\tau) \cap \alpha_I N_3 (X,\tau).
\]

Proof. Necessity. It is obvious that \( \alpha IO (X,\tau) \subseteq PIO (X,\tau) \cap \alpha_I N_3 (X,\tau) \).

Sufficiency. Let \( H \in PIO (X,\tau) \cap \alpha_I N_3 (X,\tau) \). Then we have \( H \subseteq Int(Cl^*(H)) \) and by Proposition 5, \( H = A \cap (Int(Cl^*(H)) \cup H) \) where \( A \in \alpha IO (X,\tau) \) and \( T = (Int(Cl^*(H)) \cup H) \in SIC (X,\tau) \) by Lemma 1, respectively. But

\[
T = H \cup Int(Cl^*(H)) = Int(Cl^*(H))
\]

Thus \( H = A \cap Int(Cl^*(H)) \) where \( A \in \alpha IO (X,\tau) \) and \( Int(Cl^*(H)) \in \tau \subseteq \alpha IO (X,\tau) \) and therefore \( H = A \cap Int(Cl^*(H)) \in \alpha IO (X,\tau) \), because \( \alpha IO (X,\tau) \) is a topology (see Corollary 3.2 of Acikgoz et.al [1]).

It is seen in the following example that the decomposition provided by Theorem 4 is different from the decomposition of \( \alpha-I \) continuity given in Theorem 4.2 by Acikgoz et.al [2].

**Example 3.7.** Let \( X = \{1, 2, 3, 4\} \), \( \tau = \{\emptyset, X, \{1, 2\}, \{4\}, \{1, 2, 4\}\} \) and \( I = \{\emptyset, \{3\}\} \). Then \( A = \{3\} \) is an \( \alpha_I N_3 \) set, which is not semi \( -I \) open. For, \( \text{Int}(Cl^*(A)) = \text{Int}(Cl^*(\{3\})) = \emptyset = \text{Int}(\{3\}) \), \( A = \{3\} = \{3\} \cap X \) where \( A \) is a \( t-I \) set and \( X \in \alpha IO (X) \). This shows that \( A \) is an \( \alpha_I N_3 \) set. On the other hand, since \( Cl^*(\text{Int}(A)) = \emptyset \) and \( A \nsubseteq Cl^*(\text{Int}(A)) \), \( A \) is not semi \( -I \) open.

**Proposition 6.** For a subset \( A \) of an ideal topological space \((X, \tau, I)\), the following properties are equivalent:

1. \( A \) is \( \alpha-I \) open,
2. \( A \) is pre \( -I \) open and semi \(-I \) open,
3. \( A \) is pre \( -I \) open and \( \delta-I \) open,
4. \( A \) is pre \( -I \) open and \( \alpha_I N_3 \) set.
Proof. The proof is obvious (Acikgoz et al. [2] and [1]). □

4. αI N5 - sets

**Definition 4.1.** A subset A of an ideal topological space (X, τ, I) is said to be an I-R closed [3] (resp. strong β-I-closed [8]) set if A = Cl*(Int(A)) (resp. A ⊂ Cl*(Int(Cl*(A))))

**Definition 4.2.** A subset A of an ideal topological space (X, τ, I) is said to be an AI-R-set if A ⊂ V, where U is open and V is an I-R closed set.

The family of all I-R closed (resp. strong β-I-closed) sets in an ideal topological space (X, τ, I) is denoted by IRC (X, τ) (resp. SIC (X, τ)).

**Proposition 7.** Let (X, τ, I) be an ideal topological space. A ∈ SIO (X, τ) if and only if A = R \ D where R ∈ IRC (X, τ) and Int(D) is τ*-dense.

**Proof.** Necessity. Let A ∈ SIO (X, τ). Then we have U ⊂ A ⊂ Cl*(U) such that U ∈ τ by Theorem 3.2 of [7]. Note that Cl*(A) = Cl*(U). We write A = Cl*(U) - (Cl*(U) - A) = Cl*(U) ∩ [X - (Cl*(U) - A)], where Cl*(U) - A ⊂ Cl*(U) - U and Cl*(U) - U is nowhere τ*-dense in (X, τ, I). We assume D = X - (Cl*(U) - A). Then X - Cl(Cl*(U) - A) is an open τ*-dense subset of (X, τ, I) which is contained in D. Consequently, we use R = Cl*(U) to write A = R \ D where R is an I-R closed set and Int(D) is τ*-dense, as required.

Sufficiency. Assume that A = R \ D where R is I-R closed and Int(D) is τ*-dense. We write U ∈ τ such that R = Cl*(U). We assume V = U \ Int(D). Then V ∈ τ with V ⊂ A. Finally, Cl*(V) = Cl*(U \ Int(D)) = Cl*(U) = R. Thus V ⊂ A \ Cl*(V) and therefore A ∈ SIO (X, τ) using [7]. □

**Definition 4.3.** [3]. A subset H of an ideal topological space (X, τ, I) is called an αI N5 - set if H = A \ B where A ∈ αI O (X, τ) and B is a τ*-open set.

The family of all αI N5 - sets of (X, τ, I) is denoted by αI N5 (X, τ) in this paper, when there is no chance for confusion with the ideal.

**Proposition 8.** Let (X, τ, I) be an ideal topological space and A = U \ V a subset of X. Then the following hold:

1. If A is a weakly I-locally-closed set, then A is an αI N5 - set.
2. If A is an αI N5 - set, then A is an αI N3 - set.
3. If A is an αI N3 - set, then A is δ I - open.

**Proof.** a) and b) The proof is a direct consequence of the Definition 2 and Definition 8.

c) Let A be an αI N3 - set. Then we have A = U \ V where U ∈ αI O (X, τ) and V is a t-I-set. Since every α-I-open set is δ I - open by [1] and every t-I-set is δ I - open, therefore we obtain that A ∈ δI O (X, τ). Because δI O (X, τ) is a topology (see Corollary 3.2 of Acikgoz et al. [14]). □
Remark 4.4. The converses of Proposition 3.1 need not be true as shown in the following example.

Example 4.5. Let \( X = \{1, 2, 3, 4\} \), \( \tau = \{\emptyset, X, \{1, 2\}, \{4\}, \{1, 2, 4\}\} \) and \( I = \{\emptyset, \{3\}\} \). Then \( A = \{3, 4\} \) is an \( \alpha_I N_5 \) – set, but not weakly \( I \) – locally – closed. \( \alpha IO(X) = \{\emptyset, X, \{1, 2\}, \{4\}, \{1, 2, 4\}, \{3, 4\}\} \). For a subset \( A = \{3, 4\} = X \cap \{3, 4\} \), where \( A \) is \( \alpha - I \) – open and \( X \) is \( \tau^* \) – closed. This shows that \( A \) is an \( \alpha_I N_5 \) – set. But \( A \) is not weakly \( I \) – locally – closed. Because \( A \notin \tau \).

Example 4.6. Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\} \) and \( I = \{\emptyset, \{a\}\} \). Then \( A = \{c\} \) is an \( \alpha_I N_3 \) – set but it is not an \( \alpha_I N_5 \) – set. For \( \text{Int}(\text{Cl}^*(\{c\})) = \text{Int}(\{\emptyset, \{c\}, \{a\}, \{c\}\} = \text{Int} \{\{b, c\}\} = \{c\} \), and so \( A = \emptyset \cap X \) where \( A \) is a \( \delta - I \) – set and \( X \in \alpha IO(X, \tau) \). This shows that \( A \) is an \( \alpha_I N_3 \) – set. But \( A \) is not an \( \alpha_I N_5 \) – set. For, \( \text{Cl}^*(A) = \text{Cl}^*(\{c\}) = \{\{c\}\} \cup \{c\} = \{b, c\} \neq \{c\} \) and \( \text{Cl}^*(A) \neq \emptyset \).

Example 4.7. Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}\} \), \( I = \{\emptyset, \{a\}, \{c\}, \{d\}, \{c, d\}\} \). Set \( A = \{b, d\} \). Then \( A \) is a \( \delta - I \) – open set which is not an \( \alpha_I N_3 \) – set. For \( A = \{b, d\} \), since \( \text{Cl}^*(A) = \{b, d\} \) and \( \text{Int}(\text{Cl}^*(A)) = \emptyset \) so \( \text{Int}(\text{Cl}^*(A)) \subseteq \text{Cl}^*(\text{Int}(A)) \). This shows that \( A \) is a \( \delta - I \) – open set. On the other hand, since \( A \not\subseteq \text{Int}(\text{Cl}^*(\text{Int}(A))) = \emptyset \) and \( A = A \cap X \) where \( A \notin \alpha IO(X, \tau) \), \( A \) is not an \( \alpha_I N_3 \) – set.

The two classes \( \alpha_I N_3(X, \tau) \) and \( \alpha_I N_5(X, \tau) \) are related as seen in the next proposition, whose proof is omitted since it is similar to that of Proposition 3.

Proposition 9. Let \((X, \tau, I)\) be an ideal topological space. \( H \in \alpha_I N_5(X, \tau) \) if and only if \( H = B \cap D \) where \( B \) is a weakly \( I \) – locally – closed set and \( \text{Int}(D) \) is \( \tau^* \) – dense.

Theorem 4.8. Let \((X, \tau, I)\) be an ideal topological space.

\[
\text{SIO}(X, \tau) = \text{S}\beta IO(X, \tau) \cap \alpha_I N_5(X, \tau).
\]

Proof. Necessity. Let \( A \in \text{SIO}(X, \tau) \). Then we have \( A \in \text{S}\beta IO(X, \tau) \) by Remark 1.1 of \cite{2}. Now, by Proposition 7 we write \( A = R \cap D \) where \( R \) is \( I - R \) closed and \( \text{Int}(D) \) is \( \tau^* \) – dense. Since \( R \) is a weakly \( I \) – locally – closed set ( because \( R \) is \( \tau^* \) – closed ) then \( A \) is an \( \alpha_I N_5 \) – set by Proposition 9.

Sufficiency. Let \( H \in \text{S}\beta IO(X, \tau) \cap \alpha_I N_5(X, \tau) \). Then we have \( H \subseteq \text{Cl}^*(\text{Int}(\text{Cl}^*(H))) \) and \( H = A \cap F \) where \( A \in \alpha IO(X, \tau) \) and \( F \) is \( \tau^* \) – closed, respectively. Since \( H \subseteq F \) then \( \text{Cl}^*(\text{Int}(\text{Cl}^*(H))) \subseteq \text{Cl}^*(\text{Int}(\text{Cl}^*(F))) = \text{Cl}^*(\text{Int}(F)) \subseteq \text{Cl}^*(F) = F \). Thus \( H \subseteq A \cap \text{Cl}^*(\text{Int}(\text{Cl}^*(H))) \subseteq A \cap \text{Cl}^*(\text{Int}(\text{Cl}^*(F))) \subseteq A \cap F = H \). So \( H = A \cap \text{Cl}^*(\text{Int}(\text{Cl}^*(H))) \) where \( A \in \alpha IO(X, \tau) \) and \( \text{Cl}^*(\text{Int}(\text{Cl}^*(H))) \in \text{SIO}(X, \tau) \). Thus, we have \( H \in \text{SIO}(X, \tau) \) by Proposition 2.

Proposition 10. For a subset \( A \) of an ideal topological space \((X, \tau, I)\), the following properties are equivalent:
(1) $A$ is semi–$I$–open,
(2) $A$ is strong $\beta$–$I$–open and $\delta$–$I$–open,
(3) $A$ is strong $\beta$–$I$–open and is an $\alpha_I N_5$–set.

Proof. The proof is obvious. (Acikgoz et al. [2]).

Remark 4.9. The relationships between the sets defined above, are shown in the following diagram.

DIAGRAM

Remark 4.10. By the examples stated below, we obtain the following results:

(1) $\beta$–$I$ - closedness and $\alpha_I N_3$ – set are independent of each other,
(2) $\delta$–$I$ - openness and $\beta$–$I$ - closedness are independent of each other,
(3) $t$ - $I$ - set and $\alpha_I N_5$ – set are independent of each other,
(4) Strong $\beta$–$I$ - openness and $\alpha_I N_5$ – set are independent of each other,
(5) Pre - $I$ - openness and $\alpha_I N_3$ – set are independent of each other.

Example 4.11. Let $(X, \tau, I)$ and $A$ be the same ideal topological space and the set, respectively, as in Example 1. We obtain that $A$ is $\beta$–$I$ - closed but not is not an $\alpha_I N_3$ – set. Because $A = A \cap X$ where $A$ is not a $t$ –$I$ – set and $X \in \alpha IO (X, \tau)$.

Example 4.12. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b\}\}$, $I = \{\emptyset, \{c\}\}$. Then $A = \{b\}$ is an $\alpha_I N_3$ – set which is not a $\beta$–$I$ - closed set. For, $A = \{b\} = \{b\} \cap X$ where $\{b\} \in \alpha IO (X, \tau)$ and $\text{Int}(\text{Cl}(X)) = \text{Int}(X)$. This shows that $A$ is an $\alpha_I N_3$ – set. On the other hand, since $\text{Int}(\text{Cl}(\text{Int}(A))) = X$ and $\text{Int}(\text{Cl}(\text{Int}(A))) \not\subseteq A$, $A$ is not a $\beta$–$I$ - closed set.

Example 4.13. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$, $I = \{\emptyset, \{a\}\}$. Set $A = \{a, c\}$. Then $A$ is a $\delta$–$I$ - open set but it is not a $\beta$–$I$ - closed set. For $A = \{a, c\}$, since $\text{Int}(\text{Cl}(A)) = X$, $\text{Cl}(\text{Int}(A)) = X$ and so $\text{Int}(\text{Cl}(A)) \subseteq \text{Cl}(\text{Int}(A))$. This shows that $A$ is a $\delta$–$I$ - open set. On the other hand, since $\text{Int}(\text{Cl}(\text{Int}(A))) = X$ and $\text{Int}(\text{Cl}(\text{Int}(A))) \not\subseteq A$, $A$ is not $\beta$–$I$ - closed.

Example 4.14. Let $(X, \tau, I)$ and $A$ be the same ideal topological space and the set, respectively, as in Example 1. We obtain that $A$ is $\beta$–$I$ - closed but not $\delta$–$I$ – open.

Example 4.15. Let $(X, \tau, I)$ and $A$ be the same ideal topological space and the set, respectively, as in Example 7. We obtain that $A$ is an $\alpha_I N_5$ – set but not $t$ –$I$ – set.

Example 4.16. Let $(X, \tau, I)$ and $A$ be the same ideal topological space and the set, respectively, as in Example 5. We obtain that $A$ is a $t$ –$I$ – set but not an $\alpha_I N_5$ – set.
Example 4.17. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b, c\}\}$, $I = \{\emptyset, \{c\}\}$. Set $A = \{c, d\}$. Then $A$ is an $\alpha_I N_5$ – set but it is not a strong $\beta - I$ - open set. For $A=\{c, d\}$, since $\text{Cl}^*(A) = \{c,d\}$ and $\text{Cl}^*(A) = A$, so $A = A \cap X$ where $X \in \alpha IO (X, \tau)$ and $A$ is $\tau^*$ - closed. This shows that $A$ is an $\alpha_I N_5$ – set. On the other hand, since $A \not\subset \text{Cl}^*(\text{Int}(\text{Cl}^*(A)))$, $A$ is not strong $\beta - I$ - open.

Example 4.18. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, $I = \{\emptyset, \{c\}\}$. Set $A = \{a, c\}$. Then $A$ is a strong $\beta - I$ - open set but it is not an $\alpha_I N_5$ – set. For $A = \{a, c\}$, since $\text{Cl}^*(\text{Int}(\text{Cl}^*(A))) = X$ and $A \subset \text{Cl}^*(\text{Int}(\text{Cl}^*(A)))$, $A$ is strong $\beta - I$ - open. On the other hand, since $\text{Int}(\text{Cl}^*(\text{Int}(A))) = \emptyset$ and $A \not\subset \text{Int}(\text{Cl}^*(\text{Int}(A)))$, $A$ is not an $\alpha_I N_5$ – set.

Example 4.19. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, $I = \{\emptyset, \{c\}\}$. Set $A = \{b, c\}$. Then $A$ is a pre $\gamma - I$ - open set but it is not an $\alpha_I N_3$ – set. For $A = \{b, c\}$, since $\text{Cl}^*(A) = X$ and $\text{Int}(\text{Cl}^*(A)) = X$, so $A \subset X = \text{Int}(\text{Cl}^*(A))$. This shows that $A$ is a pre $\gamma - I$ - open set. On the other hand, since $\text{Int}(\text{Cl}^*(\text{Int}(A))) = \emptyset$ and $A \not\subset \text{Int}(\text{Cl}^*(\text{Int}(A)))$, $A$ is not an $\alpha_I N_3$ – set.

Example 4.20. Let $(X, \tau, I)$ and $A$ be the same ideal topological space and the set, respectively, as in Example 5. We obtain that $A$ is an $\alpha_I N_3$ – set but it is not a pre $\gamma - I$ - open set.

5. Decompositions of $\alpha - I$ - continuity and semi - $I$ - continuity

Definition 5.1. A function $f : (X, \tau, I) \to (Y, \varphi)$ is said to be $\alpha - I$ - continuous [6] (resp. semi - $I$ - continuous [6], pre – $I$ – continuous [4], semi $\delta - I$ - continuous [2], strong $\beta - I$ - continuous [8]), if for every $V \in \varphi$, $f^{-1}(V)$ is an $\alpha - I$ - open set (resp. semi - $I$ - open set, pre – $I$ – open, semi $\delta - I$ - open, strong $\beta - I$ - open set) of $(X, \tau, I)$.

Definition 5.2. A function $f : (X, \tau, I) \to (Y, \varphi)$ is said to be $\alpha_I N_3$ – continuous (resp. $\alpha_I N_5$ – continuous) if for every $V \in \varphi$, $f^{-1}(V)$ is an $\alpha - I$ - open set (resp. $\alpha_I N_5$ – set) of $(X, \tau, I)$.

Theorem 5.3. A function $f : (X, \tau, I) \to (Y, \varphi)$ is $\alpha - I$ - continuous if and only if it is pre – $I$ – continuous and $\alpha_I N_3$ – continuous.

Proof. This is a direct consequence of Theorem 4. \qed

Theorem 5.4. For a function $f : (X, \tau, I) \to (Y, \varphi)$ the following properties are equivalent:

1. $f$ is $\alpha - I$ - continuous;
2. $f$ is pre – $I$ – continuous and semi – $I$ – continuous;
3. $f$ is pre – $I$ – continuous and $\delta - I$ - continuous;
4. $f$ is pre – $I$ – continuous and $\alpha_I N_3$ – continuous.

Proof. This follows immediately from Proposition 6. \qed
Theorem 5.5. A function \( f : (X, \tau, I) \rightarrow (Y, \varphi) \) is semi \(-I\)-continuous if and only if it is strong \( \beta - I\)-continuous and \( \alpha_I N_5 \)-continuous.

Proof. This is a direct consequence of Theorem 5. \( \square \)

Theorem 5.6. For a function \( f : (X, \tau, I) \rightarrow (Y, \varphi) \) the following properties are equivalent:

1. \( f \) is semi \(-I\)-continuous;
2. \( f \) is strong \( \beta - I\)-continuous and \( \delta - I\)-continuous;
3. \( f \) is strong \( \beta - I\)-continuous and \( \alpha_I N_5 \)-continuous.

Proof. This is an immediate consequence of Proposition 10. \( \square \)

Ozet: Bu \( \text{çalısma} \), \( \alpha_I N_3 - [3] \) \( \text{and} \) \( \alpha_I N_5 - [3] \) \( \text{kümeler} \) \( \text{verilecek,} \)
\( \alpha - I\)-açık, semi \(-I\)-açık, \( \alpha_I N_3 \) \( \text{ve} \) \( \alpha_I N_5 \) \( \text{kümelerinin} \)
\( \text{karakterizasyonları} \) \( \text{inceleenecektir.} \) \( \text{Bu kümelerden} \) \( \text{yararlanarak} \) \( \alpha \)
\(-I\)-sürekli ve semi \(-I\)-süreklinin \( \text{yeni ayrışmaları} \) \( \text{da elde} \)
edilecektir.

References

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