Algebraic Structure Of The Continuum And The Factorial Representation

by

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INTRODUCTION

In § 1 of this paper the Method of Factorial Representation is reviewed. § 2 deals with the concept of the Abelean Cluster $A_n$ and the Sequence of Abelean Clusters:

$A_1 \{0\}, A_2 \{1/2\}, A_3 \{1/6, 1/3, 2/3, 5/6\}, \ldots \ldots$

The element

$x = \sum_{k=1}^{n} a_k e_k$

of $A_n$ is characterized by the restrictions (1.6) and (1.7) of which the latter leads to

$A_m \cap A_n = \emptyset$, for $m \neq n$.

The Sequence of Clusters represents a different version of the Cantor ordering of the rational numbers. § 3 contains the real and complex universe. § 4 deals with the minimal and maximal representatives of the Abelean Clusters and the Univers. § 5 is reserved for the approximation of the irrational numbers. § 6 contains the definition of the extended universe and a criterion for transcendence related to the matrix $T \infty = (a_m)_n$ of infinite rank.

In § 7 certain Diophantine Systems are formulated, which are of special interest in connection with the factorial representation. In § 8, Alef and C are symbolically described as additive infinity: $\text{Alef} = \Sigma N (l)$ and multiplicative infinity $C = \infty!$. In § 9 the factorial representation is being brought in relation with Dedekind Cut.
§ 1

This paragraph contains certain results, which were previously analyzed in a recent paper of the author\textsuperscript{1}).

Throughout this article:

\[ e_{\lambda} = \frac{1}{\lambda!} \]

A. Every complex number \( z = x + iy \) satisfying

(1.1) \quad 0 \leq |z| \leq 1

has a unique factorial representation of the form

(1.2) \quad z = x + iy = \sum_{\lambda=1}^{\infty} (a_{\lambda} + ib_{\lambda}) e_{\lambda}

whereby \( a_{\lambda} \)'s as well as \( b_{\lambda} \)'s are integers subject to

(1.3) \quad 0 \leq a_{\lambda} \leq \lambda - 1 \text{ and } 0 \leq b_{\lambda} \leq \lambda - 1.

B. Every rational number

\( x = \frac{p}{q} \) \quad with \quad \left[ p, q \right] = 1

and subject to

(1.4) \quad 0 \leq x < 1

has a unique finite representation of the form

(1.5) \quad x = \frac{p}{q} = \sum_{\lambda=1}^{l} a_{\lambda} e_{\lambda}

where \( l = l(q) \) is obtained by determining the least factorial multiple \( l! \) of \( q \) and \( a_{\lambda} \)'s are integers satisfying

(1.6) \quad 0 \leq a_{\lambda} \leq \lambda - 1 \quad \text{for } 1 \leq \lambda \leq l - 1

and

(1.7) \quad 0 < a_{l} \leq l - 1 \quad \text{for } \lambda = l.

C. Any finite representation of the form (1.5) corresponds to a rational number \( x \) subject to (1.4).
D. The above defined function

\[ l = l(q) \]

has the following elementary properties

\[ l = l(q) = q \]

only for \( q \) prime or \( q = 4 \).

\[ l(q) \leq \max l(p_{\mu}^{x_{\mu}}) \leq q \]

for \( q = \prod_{\mu=1}^{m} p_{\mu}^{x_{\mu}} \).

E. The inverse \( l^{-1}(q) \) of \( (q) \) — with the exception of the cases stated in (1.8) — is multiple-valued.

We denote the number of integers \( q \) corresponding to the same \( l \), in other words the degree of relative multiplicity of \( l^{-1}(q) \) with

\[ I = I(l). \]

This concept plays an important role for the measurement of the density of the prime distribution.

§ 2

ABELEAN CLUSTERS

We call the set \( A_l \) consisting of elements of the form

\[ x = \sum_{\lambda=1}^{l} a_{\lambda} e_{\lambda} \]

as defined by (1.5) and subject to (1.6) and (1.7) an Abelean cluster or Simplexoid of dimension \( l \).

A. Every \( A_l \) contains

\[ N(l) = [ (l - 1)! ] \cdot (l - 1) \]

elements.

The statement A. follows from (1.5), (1.6), (1.7) immediately.

B. \( A_m \cap A_n = \emptyset \) for \( m \neq n \)

A result which follows from (1.7).
C. Every rational number
\[ x = \frac{p}{q}, \quad [p, q] = 1, \quad 0 \leq x < 1 \]
is element of only one \( A_l \) with \( l = l(q) \).
The statement C. is a simple conclusion drawn from B.

We shall refer to an abelian cluster \( A_l \) whose elements are ordered according to their magnitude as an ordered abelian cluster \( A_l^* \).

In connection of B. and C. and with respect to this last definition we can now express:

D. The Sequence
\[(2.2) \quad S : \quad A_1^*, A_2^*, \ldots, A_l^*, \ldots \]
represents a different version of the Cantor ordering of the rational elements of the unit interval.

§ 3

The Real And Complex Universe.

We call the set of elements of the form
\[ x = \sum_{\lambda=1}^{\infty} a_\lambda e_\lambda \]
where \( a_\lambda \)'s are integers subject to
\[ 0 \leq a_\lambda \leq \lambda - 1 \]

the Real Universe \( A_\infty \) and consequently the set of elements of \( a \) the form (1.2) and subject to (1.3) the complex Universe \( K_\infty \).
Obviously
\[ A_\infty \subset K_\infty \]

It is easy to make the following observations concerning \( A_l \) and \( A_\infty \).

A. \[ A_l \subseteq A_\infty \]

B. \[ S = \bigcup_{\lambda=1}^{l} A_\lambda \]
is an abelian group under addition mod. 1.

C. \( A_\infty = \bigcup_{\lambda=1}^{\infty} A_\lambda \) and \( A\infty = \bar{A}\infty \).

D. \( A\infty \) is an abelian group addition mod. 1.

E. \( A\infty \) is isomorphic to the closed unit interval.

These remarks can organically be extended to \( K_t \) and \( K\infty \).

§ 4

The Minimal and the Maximal axis of \( A_t \) and \( A\infty \).

We call

\[
(4.1) \quad m_l = \sum_{\lambda=1}^{l} e_\lambda
\]

the minimal axis

\[
(4.2) \quad M_l = \sum_{\lambda=1}^{l} (\lambda - 1) e_\lambda
\]

the maximal axis of \( A_t \), and

\[
(4.3) \quad m_\infty = \sum_{\lambda=1}^{\infty} e_\lambda = (e - 2)
\]

minimal axis and

\[
(4.4) \quad M_\infty = \sum_{\lambda=1}^{\infty} (\lambda - 1) e_\lambda = 1
\]

the maximal axis of the real universe \( A\infty \) respectively.

We call \( R_t \) defined by

\[
(4.5) \quad M_\infty - M_l = R_t = \sum_{\lambda=l+1}^{\infty} (\lambda - 1) e_\lambda
\]

the residua complement of \( M \).

Due to (4.4) and the identity

\[
(4.6) \quad M_l + e_l = 1
\]
we observe elementarily the following property:

A.

\[ e_l = R_l \]  

From the geometric point of view this result means that \( e_l \) is a maximal element of the residual space \( R_l \).

§ 5

*Approximation of the Irrational Numbers*

We call \( x \in A \) with infinitely many non-zero components a proper element of \( A \).

A. According to \( A, B, C \), of § 1 with the exclusion of \( x = 1 \) all the other proper elements of \( A \) are irrational numbers, satisfying \( 0 \leq x \leq 1 \).

B. For every proper element \( x \) of \( A \) the following inequality holds:

\[ x = \sum_{\lambda=1}^{n} a_\lambda e_\lambda + \sum_{\lambda=n+1}^{\infty} a_\lambda e_\lambda \leq \sum_{\lambda=1}^{n} a_\lambda e_\lambda + R_n = x_n + R_n. \]

Hence:

C.

\[ x - x_n \leq R_n \]

or according to (4.7)

\[ x - x_n \leq e_n. \]

We call

\[ x_n = \sum_{\lambda=1}^{n} a_\lambda e_\lambda \]

of (5.1) the proper \( n \)-th representative of the irrational number \( x \).

D. Due to (1.6), \( x_n \) is the largest element of \( A_n \) satisfying

\[ 0 < x_n < x. \]
The concept proper $n$-th representative can also be applied for

\begin{equation}
\sum_{\lambda=1}^{l} a_{\lambda} e_{\lambda}, \quad y \in A_{l}.
\end{equation}

For $n < l$ we call

\begin{equation}
\sum_{\lambda=1}^{n} a_{\lambda} e_{\lambda}
\end{equation}
correspondingly the proper $n$-th representative of $y$. (5.3) remains also valid for this interpretation.

\section{6}

The extended Universe and The Criterion of Transcendence

We call the sequence

\begin{equation}
E: \ e_{1}, \ e_{2}, \ldots, \ e_{n}, \ldots
\end{equation}

the fundamental base of the Universe and the sequence

\begin{equation}
Y: \ y_{1}, \ y_{2}, \ldots, \ y_{n}, \ldots
\end{equation}

introduced by

\begin{equation}
\sum_{\lambda=1}^{\infty} b_{n\lambda} e_{\lambda} \quad \text{for} \ n = 1, 2, \ldots
\end{equation}

the regular base of the Extenden Universe $U_\infty$, provided the infinite matrix of the transformation

\begin{equation}
T_{\infty} = (b_{n\lambda}), \ (n, \ \lambda) = 1, 2, \ldots
\end{equation}
is of infinite rank and the components are finite complex numbers. Obviously:

A.

\[ \Lambda_\infty \subset K_\infty \subset U_\infty. \]

We refer to $x$ as a transcendental element of $U_\infty$, whenever the power sequence

\begin{equation}
P: \ x, \ x^2, \ldots, \ x^n, \ldots
\end{equation}
is a regular base of $U_\infty$. 
Example 1. The power sequence

(6.6) \[ E_1 : e^m = \sum_{\lambda=0}^{\infty} m^\lambda \ e_\lambda \]  

with the infinite matrix:

(6.7) \[ T_\infty = (m^\lambda) \]

\[ m = 1, 2, \ldots ; \lambda = 0, 1, 2, \ldots \]

is a regular base.

Example 2. The power sequence

(6.8) \[ E : (e - 2), (e - 2)^2, \ldots, (e - 2)^n, \ldots \]

is a regular base.

Remark.

(6.9) \[ (e - 2)^n = \sum_{\lambda=0}^{n} (\lambda^n) (e - 2)^{n-\lambda} \ e^\lambda \]

is a linear combination of

(6.10) \[ 1, \ e^1, \ldots, \ e^n \]

B. If \( x \) is transcendental with respect to \( U_\infty \) then \( x - [x] \)

is transcendental with respect to \( \Lambda \). \([x]\) is the integral part of \( x \).

We call \((e_\lambda z^\lambda) - c\) and \( z \) complex — local contraction

and \( e_\lambda \) the \( \lambda \) — th contractor.

C. If \( f(z) = \sum_{\lambda=0}^{\infty} c_\lambda \ z^\lambda \ e_\lambda \)

is analytic over \[ |z| < R \leq \infty \]

then \( f(z) \in U_\infty \).

§ 7

Certain Diophantine Systems

A close study of linear independence of the fundamental and regular bases leads to certain mathematical questions, which are attractive for their own sake.
The following Diophantine System is a typical example of this kind:

\[(7.1) \quad s \cdot (\lambda - 1) = \sum_{k=1}^{m} t_k \cdot a_{k\lambda} \]

\[\lambda = 1, 2, \ldots, n, \ldots\]

where \(t_k\) s are integers and \(a_{k\lambda}\)'s are integers subject to

\[(7.2) \quad 0 \leq a_{k\lambda} \leq \lambda - 1\]

\[(7.3) \quad m \leq \infty .\]

**Example 1.** \(x = \sum_{\lambda=1}^{\infty} a_{1\lambda} e_\lambda \in A_\infty, \lambda = \sum_{\lambda=1}^{\infty} a_{2\lambda} e_\lambda \in A_\infty\)

and \(x + y = 1\)

\(s=1, t_1 = t_2 = 1, m = 2\) and \(a_{1\lambda} + a_{2\lambda} = \lambda - 1 .\)

**Example 2.** For \(e - 2 = x_n + r_n, n = 1, 2, \ldots\) the residual sequence

\(r_1, r_2, \ldots, r_n, \ldots\)

is a regular base and

\(t_k = 1\) for \(k = 1, 2, \ldots, n, \ldots\)

\(a_{k\lambda} = 1\) for \(\lambda \geq k\) and \(a_{k\lambda} = 0\) for \(\lambda < k .\)

\[\S 8\]

**Symbolic Interpretation of Alef and C**

**A. The power of the countable set**

\[(8.1) \quad \text{Alef} = \sum_{\lambda=1}^{\infty} N(\lambda) \text{ (additive infinity)}\]

and in connection with (1.2) and (1.3).

**B. The power of the Continuum**

\[(8.2) \quad \text{C} = 1.2.3 \ldots n \ldots = \infty \text{ (multiplicative infinity).}\]
§ 9

The Dedekind Cut and Factorial Representation

For every $x \in A_\infty$, we use the finite or infinite sequence

$$(9.1) \quad A_{k_1}, A_{k_2}, \ldots, A_{k_n}, \ldots$$

which is obtained from the sequence

$$(9.2) \quad A_1, A_2, \ldots, A_n, \ldots$$

by the omission of those clusters $A_n$ in which there is no-representative of $x$, The Reduced Sequence of the Representatives of $x$. Obviously (9.1) might as well be identical with (9.2).

A. Every $x \in A_\infty$ is a Dedekind Cut represented by the finite or infinite sequence

$$(9.3) \quad x_{k_1}, x_{k_2}, \ldots, x_{k_n}, \ldots$$

$x_{k_n} \in A_\infty$ and maximal relative to $x$.

B. $x_{k_n} = \varnothing_x (A_{k_n})$ is the relative choice function of Zermelo.

C. Every subset of (9.3) has a maximal element, $x$ is a maximal element (Zorin's Lemma).

REFERENCES


x2e On the suggestion of the author Ulaş Çkapar of M.E.T.U has made an interesting survey concerning the nature of $1 (q)$ in 1967 (unpublished). His observations are not included.

x3e On the suggestion of the author Ellen Fenwick of Rutgers (with the help of Robert Turkelson of Rutgers) has conducted a programed survey concerning the related distribution of $l^{-1} (g)$, in 1968. The method they used is different in nature from the one applied in the previous paper of the author, (unpublished).
ÖZET

Continuum’un Cebirsel Yapısı ve Faktöryel Temsil

Bu makale aslında bundan önce Orta Doğu Teknik Üniversitesi Temel ve Uygulamalı Bilimler Dergisi Cilt 1, Sayı 2, yıl 1968 Sayfa 71 de yayınlanan bir araştırmannın devamıdır. § 1 de temel konuya değinilmiştir. § 2 de Abel Kümleri tanımlanmıştır. Düzenlenmiş Abel Kümleri yardımıyla rasyonel sayıların sıralanabilirliği Cantor metodundan farklı bir şekilde gösterilmiştir. § 3 de reel ve kompleks universunun tanımlanmış, § 4 te Abel Kümlerinin ve universunın minimal ve maximal eksenleri ifade olunmuştur. § 5 irrasyonel sayıların yaklaşı değerlerinin hesaplanması ve hata tahminine dairdir. § 6 da genelleştirilmiş universunun tanımlanmakta ve bir transcendence kriteri verilmektedir. Aynı paragrafta e ve (e-2) örnekleri incelenmektedir. § 7 de faktöryel temsil ile bazı Diophant denklem sistemleri ve o tipten iki örnek andelenmektedir. § 8 de sembolik olarak Alef = Σ N (0), C = ∞ ! eşitlikleri ileri sürülmektedir. § 9 da faktöryel temsil Dedekind kesiti ile karşılaştırılmaktadır.
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