Matrix Transformations and Generalized Almost Convergence

by

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Matrix Transformations and Generalized Almost Convergence

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ABSTRACT

Recently M. Stieglitz introduced the concept of $F_{\beta}$-convergence of sequence and generalized the results of Eizen and Laush, King, and Shaefer for more general classes of matrices. Quite recently Ahmad and Mursaleen have extended the space $F_{\beta}$ of $F_{\beta}$-convergent sequences to $F_{\beta}(p)$. In the present paper we furnish a set of necessary and sufficient conditions for each $(c_{\circ}(p), F_{o\beta}(p)), (1(p), F_{\beta})$ and $(M_{a}(p), F_{\beta})$ matrices.

1. INTRODUCTION

In 1948, Lorentz [2] introduced the concept of almost convergence. In 1973, recently M. Stieglitz [6] generalized this concept of almost convergence to $F_{\beta}$-convergence. And quite recently Ahmad and Mursaleen [1] extended this space $F_{\beta}$ of $F_{\beta}$-convergent sequences to $F_{\beta}(p)$ just as $c, c_{\circ}$ and $f$ were extended to $c(p), c_{\circ}(p)$ and $f(p)$ respectively. For real $p_{n} > 0$ and sup $p_{n} < \infty$, we have [1],

$F_{\beta}(p) = \{ x: \lim_{n} (B_{L}x)_{n} = 0, \text{uniformly in } i, \text{for some } L \}$

$F_{o\beta}(p) = \{ x: \lim_{n} (B_{L}x)_{n} = 0, \text{uniformly in } i \}$

In particular, if $p_{n} = p > 0$ for every $n$, we have

$F_{\beta}(p) = F_{\beta}$ and $F_{o\beta}(p) = F_{o\beta}$. If we put $\beta = \beta_{o}$

$F_{\beta}(p)$ reduces to $c(p)$ and if $\beta = \beta_{s}$, $F_{\beta}(p) = f(p)$.

2. In this note, we prove the following theorems:

Theorem 2.1. $A \in (c_{\circ}(p), F_{q\beta}(p))$ if and only if
(i) \( N(A) < \infty \) and there exist \( r \geq 0 \) and \( B > 1 \) such that

\[
\sup_{0 \leq i < \infty} \left\{ \sum_{k} b_{nk} (i) a_{kl} \mid B \right\}^{1/p_1} < \infty
\]

\( r \leq n < \infty \)

(ii) \( \lim_{n \to \infty} \left| \sum_{i \leq k} a_{ik} \right|^{p_s} = 0 \) (uniformly in \( i, k \) fixed).

Proof. Necessity. Suppose \( A \in (c_0 (p), F_\theta (p)) \). We know the fact \( A : c_0 \to m \), hence \( N(A) < \infty \). Define \( e_k = \{ \delta_{jk} \} \), where

\[
\delta_{jk} = \begin{cases} 0 & (j \neq k), \\ 1 & (j = k). \end{cases}
\]

Since \( e_k \in c_0 (p) \), now

\[
T = \sum_{i \leq k} b_{ni} (i) \sum_{k} a_{ij} \delta_{jk}
\]

\[
= \sum_{i \leq k} b_{ni} (i) a_{ik}
\]

Therefore, for all \( k \geq 1 \), \( \delta_{jk} \to 0 \) as \( j \to \infty \), it follows that

\[
\lim_{n \to \infty} \left| \sum_{i \leq k} b_{ni} (i) \sum_{k} a_{ij} \delta_{jk} \right|^{p_s} = 0
\]

This is equivalent to (ii). Now, put

\[
f_m (x) = \sum_{l=0}^{m} \left( \sum_{k} b_{nk} (i) a_{kl} \right) x_l
\]

It is easy to see that \( \{ f_m (x) \} \) is a sequence of continuous linear functionals such that \( \lim_m f_m (x) \) exists. We note that

\[
Tx = \left| \sum_{i \leq k} (B_i (Ax))_a \right|^{p_s} = \lim_m f_m (x).
\]

Therefore, by virtue of Banach-Steinhaus theorem, it follows that \( T \in c_0 (p) \) (continuous dual space of \( c_0 (p) \)) and \( \| T \| < \infty \). Let us define for each \( r \):

\[
\gamma_{1} = \begin{cases} \frac{k}{p_1} \sgn \left( \sum_{k} b_{nk} (i) a_{kl} \right) & (0 \leq l \leq r), \\ 0 & (\text{otherwise}). \end{cases}
\]
where $K$ is a constant. Then it follows that
\[
\begin{align*}
(r) & \quad y_1 \in c_1 (p) \\
\text{and} & \quad \left\{ \begin{array}{l}
\sum_{k=1}^{r} b_{nk} (i) a_{kl} -1/p_l < B \quad p_n \leq K \\
-K
\end{array} \right. 
\end{align*}
\]
for each $n$ and $r$, where $B = \delta$. Therefore (i) holds.

Sufficiency. Let us suppose that the conditions (i) and (ii) hold and that $x \in c_1 (p)$. For $C = \max (1, 2^{H-1})$ where $H = \sup p_n$, we have the inequality (see Maddox [5], p. 346)
\[
|B(Ax)_n| \leq C(I_1 + I_2)
\]
where
\[
I_1 = \sum_{l \leq l_0}^{p_n} |\sum_{k=1}^{r} b_{nk} (i) a_{kl} x_k|,
\]
and
\[
I_2 = \sum_{l > l_0}^{p_n} |\sum_{k=1}^{r} b_{nk} (i) a_{kl} x_k|.
\]
1 and $n$ both are larger than $l_0$.

Since (ii) holds, therefore, there exists $n_0 > 0$ such that $n > n_0$,
\[
|\sum_{k=1}^{r} b_{nk} (i) a_{kl} x_k| < \varepsilon, \text{ uniformly in } i.
\]
Therefore, for such $n$

(I) \quad $I_1 < (\sum_{l \leq l_0}^{p_n} |(\sum_{k=1}^{r} b_{nk} (i) a_{kl}) x_k|)$
\[
< \varepsilon (\sum_{l \leq l_0}^{p_n} x_k) \text{ uniformly in } i.
\]

Again for $n > n_0$,

(II) \quad $I_2 < (\sum_{l > l_0}^{p_n} |(\sum_{k=1}^{r} b_{nk} (i) a_{kl}) x_k|)$
\[
< \varepsilon \text{ uniformly in } i.
\]

Hence the sufficiency follows from (I) and (II).
This completes the proof.

Theorem 2.2. A $\varepsilon (l \ (p), \ F_\beta )$ if and only if

(i) There exists $B > 1$ such that for every $i$

\[
\sup_n \sum_k q_k^{q_k} C(n, k, i) B < \infty \quad (1 < p_k < \infty)
\]

\[
\sup_n p_k < \infty, (0 < p_k \leq 1)
\]

where

\[
C(n, k, i) = \Sigma l b_{nl} (i) a_{lk}
\]

(ii) \( \lim_n C(n, k, i) = a_k \) (uniformly $i, k$ fixed).

Proof. Necessity. We only consider the case $1 < p_k < \infty$.

The case $0 < p_k \leq 1$ has a similar proof. Let $A \in l \ (p), \ F_\beta$.

Define $e_k = (0, 0, ..., 0, l, 0, ..., )$. Since $e_k \in l \ (p)$, (ii) must hold.

Now $(B_1 (Ax))_n$ exists for each $n$ and $x \in l \ (p)$. If we put $T_{n+1} (x) = (B_1 (Ax))_n$, then \( \{T_{n+1} (x)\}_n \) is a sequence of continuous real functions on $l \ (p)$ and further \( \sup_n |(B_1 (Ax))_n| < \infty \) on $l \ (p)$.

Now by uniform boundedness principle (see Lascarides and Maddox [3]) the necessity follows.

Sufficiency. For every $j \geq 1$, we have

\[
\sum_{k=1}^j C(n,k,i) q_k^{q_k} B \leq \sup_n \sum_k \sum_{l=1} |b_{nl} (i) a_{lk}| q_k^{q_k} B
\]

therefore

\[
\sum_k a_k q_k^{q_k} B \leq \lim_{j \to \infty} \lim_{n \to \infty} \sum_{k=1}^j C(n,k,i) q_k^{q_k} B
\]

\[
\leq \sup_n \sum_k q_k^{q_k} B < \infty.
\]

Thus the series $\sum_k C(n,k,i) x_k$ and $\sum_k a_k x_k$ converge (see Maddox [4]) for each $n$ and $x \in l \ (p)$. Now, for $\varepsilon > 0$ and $x \in l \ (p)$, choose $k_o$ such that
\begin{align*}
\sum_{k=1}^{k_0} (C(n,k,i) - \alpha_k) &< \varepsilon \ \forall \ n > n_o.
\end{align*}

By (ii) there exists \( n_o \) such that
\[
\left| \sum_{k=1}^{k_0} (C(n,k,i) - \alpha_k) \right| < \varepsilon \ \forall \ n > n_o.
\]

Since (i) holds, it follows that (see Lascarides and Maddox [3]).
\[
\left| \sum_{k=1}^{\infty} C(n,k,i) - \alpha_k \right|
\]
is arbitrarily small. Therefore,
\[
\lim_{n} \sum_{k} C(n,k,i) \ x_k = \sum_{k} \alpha_k \ x_k
\]
uniformly in \( i \). This completes the proof.

Corollary. A \( \varepsilon (l(p), F_{b}) \) if and only if

(i) Condition (i) theorem (2.2) holds,

(ii) \( \lim_{n \to \infty} C(n,k,i) = 0 \) uniformly in \( i \).

We now characterise the matrices in the class \((M_o \ (p), F_{b})\).

For \( p_k > 0 \) we define (see Maddox [4]),
\[
M_o(p) = \{ x : \sum_{k} |x_k| B < \infty \}, \quad B > 1
\]
When \( p_k = p \ \forall \ k \), we have \( M_o \ (p) = l_1 \). Also \( M_o \ (p) = l_1 \) for \( \inf p_k > 0 \).

Theorem 2.3. A \( \varepsilon (M_o \ (p), F_{b}) \) if and only if

(i) For every integer \( B > 1 \),
\[
\sup_{n,k} C(n,k,i) |B| < \infty \ (\forall \ i)
\]

(ii) \( \lim_{n} C(n,k,i) = \alpha_k \) (uniformly in \( i, k \) fixed)

Proof. Necessity. Suppose A \( \varepsilon (M_o \ (p), F_{b}) \). Since \( e_k \varepsilon M_o \ (p) \), (ii) holds. On contrary let us suppose that (i) is not true, then \( \exists \ B > 1 \) such that
\[ \sup_{n,k} \left| C(n, k, i) \right| B^{1/p_k} = \infty \]

So by theorem (2.2), \( C = (C_{nk}) = (a_{nk} B^{1/p_k}) \neq (l_i, F_\beta) \) that is,

there exists \( x \in L_1 \) such that \( Cx \notin F_\beta \). Now \( y = (y_k) = (B x_k) \in M_o(p) \), but \( Ay = Cx \notin F_\beta \), which contradicts that \( A \in (M_o(p), F_\beta) \).

Sufficiency. Suppose that the conditions (i) and (ii) hold and \( x \in M_o(p) \). Then

\[
\left| \sum C(n, k, i) x_k \right| \leq \sum x_k \left| B^{1/p_k} \right| C(n, k, i) B^{1/p_k} < \infty.
\]

Now similarly as in Theorem (2.2) we have

\[
\lim_{n \to \infty} \sum_{k} C(n, k, i) x_k = \sum_{k} a_k x_k
\]

uniformly in \( i \) and hence \( A \in (M_o(p), F_\beta) \).

This completes the proof.

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REFERENCES


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