Matrix Transformations And Generalized Almost Convergence II

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Matrix Transformations And Generalized Almost Convergence II

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ABSTRACT

The purpose of this paper is to investigate some more classes of matrices which will fill up a gap in the existing literature. We have already characterized the \( (c_o (p), F_{\sigma B} (p)), (I (p), F_{\sigma B}) \) and \( (M_o (p), F_B) \) matrices. In the present paper author characterizes \( (w (p), F_{\sigma B}) \) \( (w (p), F_{\sigma B}) \), \( (c (p), F_B) \) \( (c (p), F_B) \) \( (l_{\infty} (p), F_B) \) \( (l_{\infty} (p), F_B) \) and \( (\hat{c} (p), F_B) \) \( (\hat{c} (p), F_B) \) matrices.

1. INTRODUCTION

Let \( l_{\infty}, c \) and \( c_o \) be the Banach spaces of bounded, convergent and null sequences \( x = \{x_k\} \) with the usual norm \( \|x\| = \sup_k |x_k| \). A sequence \( x \in l_{\infty} \) is almost convergent \([1]\) if all Banach limits of \( x \) coincide. Let \( \hat{c} \) denotes the space of almost convergent sequences. If \( p_k \) is real such that \( p_k > 0 \) and \( \sup p_k < \infty \), we define (see Maddox \([3]\), Simons \([8]\) and Nanda \([6]\))

\[
w (p) = \{x: n^{-1} \sum_{k=1}^{n} |x_k - 1|^p_k \to 0 \text{ for some } l\}
\]

\[
l_{\infty} (p) = \{x: \sup_k |x_k|^p_k < \infty\},
\]

\[
c(p) = \{x: |x_k - 1|^p_k \to 0 \text{ for some } l\},
\]

and

\[
\hat{c} (p) = \{x: \lim_k |t_{k,1} (x_l - 1 e)|^p_k = 0 \text{ for some } l, \text{ uniformly in } i\}
\]

where
when \( p_k = p \) \( \forall \) k, we have \( w(p) = w_p, l_{\infty}(p) = l_{\infty} \),
c(p) = c and \( \hat{c}(p) = \hat{c} \) respectively.

Quite recently M. Stieglitz [9] generalized almost convergence by defining \( F_\beta \) - convergence in the following manner: Given a matrix sequence \( \beta = (B_i) \) with \( B_1 = (b_{nk}(i)) \), the sequence \( x \in l_{\infty} \) is \( F_\beta \) - convergent to the value \( \text{Lim} \beta x \), if

\[
\lim_{n} (B_1 x)_n = \lim_{n} \sum_{k=0}^{\infty} b_{nk}(i) x_k = \lim \beta x \text{ (uniformly in } i) \]

holds. The space \( F_\beta \) of \( F_\beta \) - convergent sequences depends on the fixed choosen matrix \( \beta = (B_i) \), in case \( B_o = (I) \) it is equal to \( c \) and in case \( B_j = (B_i^{(j)}) \) it is equal to \( \hat{c} \).

We have already examined the classes of \( (c_o(p), F_{o\beta}(p)) \),

\( (l(p), F_\beta) \), and \( (M_o(p), F_\beta) \) - matrices (see [5]). In this paper, Theorems 2.1 and 2.2 generalize the results of Lascarides and Maddox [2] and Nanda [7]. In Theorems 3.1, 3.2 and 3.3 we determine the matrices \( (c(p), F_\beta), (l_{\infty}(p), F_\beta) \) and \( (\hat{c}(p), F_\beta) \) which generalize the results of Stieglitz [9].

2. We prove the following Theorems

Theorem 2.1. Let \( o < p_k \leq 1 \), then \( A \in (w(p), F_\beta) \), if and only if

(i) There exist \( B > 1 \) such that

\[
Q_1 = \sup_{n} \max_{r=o}^{\infty} (2^r B^{-1})^{1/p_k} | c(n, k, i) | < \infty (\forall i)
\]

(ii) \( \lim_{n} c(n, k, i) = \alpha_k \) uniformly in \( i, k \) fixed,

(iii) \( \lim_{n} \sum_{k} c(n, k, i) = \alpha \) uniformly in \( i \).

where

\[
c(n, k, i) = \sum_{j} b_{nj}(i) a_{jk}
\]
Proof. Necessity. Suppose that $\Lambda \in (w(p), F_{\overline{B}^p})$. Since $e_k$ and $e$ are in $w(p)$, (ii) and (iii) must hold, where $e_k = \{0, 0, \ldots, 0, 1, 0, 0, \ldots\}$ and $e = \{1, 1, 1, \ldots\}$.

Now $\sum_k c(n, k, i) x_k$ converges for each $n$ and $x \in w(p)$.

Therefore $(c(n, k, i))_k \in w(p)^+$ and

$$\sum_{k=0}^{\infty} \max_{r=0}^{1/p} (2 \overline{B})^{1/k} | c(n, k, i) | < \infty,$$

for each $n$ (see Lascarides and Maddox [2]).

Further, denote $\sigma_{n,i}(x) = T_{n,i}(Ax) = \sum_k c(n, k, i) x_k$,

then $\{\sigma_{n,i}\}$ is a sequence of continuous linear functionals on $w(p)$ such that $\lim_{n} T_{n,i}(Ax)$ exists. Therefore by Banach – Steinhaus Theorem [4], (i) holds.

Sufficiency. Suppose that the conditions (i) — (iii) hold. Then $(c(n, k, i))$ and $(\alpha_k)$ are in $w(p)^+$ (see [2]).

Therefore the series $\sum_k c(n, k, i)x_k$ and $\sum_k \alpha_k x_k$ converge for each $n$ and $x \in w(p)$. Put

$$f(n, k, i) = c(n, k, i) - \alpha_k.$$

Therefore

$$\sum_k c(n, k, i)x_k = \sum_k \alpha_k x_k + \sum_k f(n, k, i)(x_k - 1)$$

where $l = \lim_{n} x_k$. By (ii) we have

$$\lim_{n} \sum_{k \geq k_0} f(n, k, i)(x_k - 1) = 0.$$

Also since

$$\sup_{n} \max_{r=0}^{1/p} (2 \overline{B})^{1/k} | f(n, k, i) | \leq 2 Q_i,$$

$$\lim_{n} \sum_{k \geq k_0} |f(n, k,i) | x_k - 1 | = 0.$$

Hence
\[
\lim \sum_{n, k, i} c(n, k, i) x_k = 1 + \sum_k \alpha_k (x_k - 1)
\]
and therefore proof is complete.

**Theorem 2.2 (a).** Let \(1 \leq p < \infty\), then \(A \in (w_p, F_{FB})\) if and only if

(i) \(M = \sup_{n \to -\infty} \sum_{r} 2^{r/p} T_r^p (n, i) < \infty , \ (\forall i)\)

(ii) \(\lim_{n} c(n, k, i) = \alpha_k\) uniformly in \(i, k\) fixed

(iii) \(\lim_{n} \sum_{k} c(n, k, i) = \alpha\) uniformly in \(i\).

where

\[T_r^p (n, i) = (\sum_r |c(n, k, i)|^q)^{1/q} (p^{-1} + q^{-1} = 1).\]

(the summation is taken over \(k\) with \(2^r \leq k < 2^{r+1}\)).

(b). Let \(0 < p < \infty\). Then \(A \in (w_p, F_{FB})_{reg}\) if and only if conditions (i), (ii) with \(\alpha_k = 0\) and (iii) with \(\alpha = 1\) hold.

**Proof (a).** **Necessity.** Suppose that \(A \in (w_p, F_{FB})\). Since \(c_k\) and \(e\) are in \(w_p\), therefore, (ii) and (iii) must hold. Now define for each \(n\) and \(r \geq 0\), \(g_{r,n} (x) = \sum c(n, k, i) x_k\). Sequence \(\{g_{r,n}\}\) is of continuous linear functional in \(w_p\). Now

\[|g_{r,n} (x)| \leq (\sum_r |c(n, k, i)|^q)^{1/q} (\sum_r |x_k|^p)^{1/p}\]

\[\leq 2^{r/p} T_r^p (n, i) \|x\|\]

and

\[\lim_{n \to -\infty} \sum_{r=0}^\infty g_{r,n} (x) = T_{n,1} (Ax) < \infty\]

Therefore by Banach – Steinhaus Theorem there exists \(K\) such that

\[|T_{n,1} (A x)| \leq K \|x\|\]

Since \(l\) is arbitrary and if we define \(x \in w_p\) as in Maddox ([4], Theorem 7) we have

\[\sum_{r=0}^\infty 2^{r/p} T_r^p (n, i) \leq K\]
Therefore by the same argument as in Theorem (2.1) we see that (i) holds.

**Sufficiency.** Let us suppose that the conditions (i) – (iii) be satisfied and \( x \in w_p \). Since

\[
| T_{n,i} (A \, x) | \leq \sum_{r=0}^{\infty} \sum_r \left| c(n,k,i) x_k \right|^{1/q} \left( \sum_r | x_k |^p \right)^{1/p}
\]

\[
\leq \sum_{r=0}^{\infty} \left( \sum_r \left| c(n,k,i) \right|^q \right)^{1/q} \left( \sum_r | x_k |^p \right)^{1/p}
\]

\[
\leq M \| x \|.
\]

Therefore \( T_{n,i} (A \, x) \) is absolutely and uniformly convergent for each \( n \). Since

\[
\sum_{r=0}^{\infty} 2^{r/p} \left( \sum_r \left| \alpha_k \right|^q \right)^{1/q} < \infty \text{ and } \sum_r \alpha_k x_k < \infty.
\]

Therefore as in Theorem (2.1), \( A \in (w_p, F_{\bar{B}}) \). Which completes the proof.

Proof of (b) is constructed from the proof of (a).

3. **Some further Results**

**Theorem 3.1** (a). \( A \in (c\, (p), F_{\bar{B}}) \) if and only if

(i) There exists an integer \( B > 1 \) such that

\[
G_i = \sup_n \sum_k \left| c(n,k,i) \right| B^{\frac{k}{p}} < \infty, \quad (\forall \, i)
\]

(ii) \( \lim_{n} c(n,k,i) = \alpha_k \), uniformly in \( i, k \) fixed

(iii) \( \lim_n \sum_k c(n,k,i) = \alpha \), uniformly in \( i,\)

where

\[
c(n,k,i) = \sum_j b_{nj} (i) a_{jk}
\]

(b) \( A \in (c_0\, (p), F_{\bar{B}}) \) if and only if conditions (i) and (ii) of Theorem (a) holds.
(c) \( A \in (c(p), F_{\mathcal{B}})_{\text{reg}} \) if and only if conditions (i), (ii) with \( \alpha_k = 0 \) and (iii) with \( \alpha = 1 \) hold.

Proof (a) Necessity. Let \( A \in (c(p), F_{\mathcal{B}}) \). Define \( e = (1, 1, \ldots) \) and \( e_k = (0, 0, 0, 1, 0, \ldots) \). Since \( e \) and \( e_k \) are in \( c(p) \), (ii) and (iii) must hold. Put \( \sigma_{n_1}(x) = T_{n_1}(Ax) = \Sigma_k c(n,k,i)x_k \). Since \( (c(p), F_{\mathcal{B}}) \subset (c_0(p), F_{\mathcal{B}}) \), \( \{ \sigma_{n_1} \} \) is a sequence of continuous linear functionals on \( c_0(p) \), such that \( \lim_{n} \sigma_{n_1}(x) \) exists uniformly in \( i \). Therefore by uniform boundedness principle for \( 0 < \delta < 1 \), there exists a constant \( K \) such that \( \sigma_{n_1}(x) \leq K \) for each \( n \) and \( x \in c(p) \). Let us define \( x^r = (x^r_k) \in c(p) \) by the following:

\[
x^r_k = \begin{cases} \delta^{K/p_k} \text{sgn}(c(n,k,i)), & 0 \leq k \leq r; \\ 0, & r < k. \end{cases}
\]

Then, it follows that

\[
\sum_{k=0}^{r} |c(n,k,i)| B^{-l/p_k} \leq K
\]

for each \( n \) and \( r \), where \( B = \delta^{-K} \). Therefore (i) holds.

Sufficiency. Suppose that the conditions (i) — (iii) hold and \( x \in c(p) \). Then there exists \( l \) such that

\[
| x_k - 1 |^{p_k} \to 0. \text{ Hence for a given } \epsilon > 0, \text{ there exists an integer } k_o \text{ such that } \forall k_o > k
\]

\[
| x_k - 1 |^{p_k/M} \leq \frac{\epsilon}{B(2G_1 + 1)} < 1
\]

and therefore for \( k_o > k \)

\[
B^{1/p_k} | x_k - 1 | < B^{M/p_k} x_k - 1 | \leq \left( \frac{\epsilon}{2G_1 + 1} \right)^{M/p_k} \leq \frac{\epsilon}{2G_1 + 1}.
\]

By (i) and (ii) we have
\[ \sum_{k} c(n, k, i) - \alpha_k \mid B^{-1/p_k} < 2 G_i. \]

Hence
\[ \sum_{k > k_o} \mid (c(n, k, i) - \alpha_k)(x_k - 1) \mid < \varepsilon. \]

Also
\[ \lim_{n} \sum_{k \leq k_o} \mid (c(n, k, i) - \alpha_k)(x_k - 1) \mid = 0 \]

uniformly in i. Therefore combining the above facts we have
\[ \lim_{n} \sum_{k} c(n, k, i)x_k = 1 + \sum_{k} \alpha_k(x_k - 1) \]

uniformly in i. This proves that \( A \in (c(p), F_{
abla B}) \).

(b) Since \( x \in c_o(p) \Rightarrow 1 = 0 \), therefore the proof is immediate.

(c) First we observe that \( \alpha_k = 0 \) and \( \alpha = 1 \), proof follows immediately.

**Theorem 3.2.** (a) \( A \in (l_\infty(p), F_{\nabla B}) \) if and only if

(i) \( \lim_{n} c(n, k, i) = \alpha_k \) uniformly in i, k fixed.

(ii) \( \sup_{n} \sum_{k} c(n, k, i) < \infty \quad (\forall i) \)

(iii) There exists an integer \( N > 1 \) such that

\[ \lim_{n} \sum_{k} \mid c(n, k, i) - \alpha_k \mid N^{1/p_k} = 0 \] uniformly in i.

(b) \( A \in (l_\infty(p), F_{\nabla B}) \) iff (i) condition (ii) of Theorem (a) holds, (ii)
lim \[ \sum_{n} \mid c(n, k, i) \mid N^{1/p_k} = 0 \] uniformly in i.

**Proof (a). Necessity.** Suppose that \( A \in (l_\infty(p), F_{\nabla B}) \), since \( e_k \in l_\infty(p) \), (i) must hold. Since \( (l_\infty(p), F_{\nabla B}) \) (c, \( F_{\nabla B} \))
(ii) holds. If (iii) is not true then the matrix
\[ C = (c_{nk}) = (a_{nk} N^{1/p_k}) \notin (l_\infty, F_{\nabla B}) \] for some integer \( N > 1 \). So that there exists \( x \in l_\infty \).
such that \( B \times \not\in F_{\bar{B}}. \) Now \( y = (y_k) = (N^{1/p_k} x_k) \in l_\infty (p), \)
but \( Ay = C \times \not\in F_{\bar{B}}. \) This contradicts the fact that \( A \in (l_\infty (p), F_{\bar{B}}). \) Hence (iii) is true.

**Sufficiency.** Suppose that the conditions (i) -- (iii) hold. Choose an integer \( N > \max (1, \sup_k | x_k |^{p_k}) \). By (ii)
\[
| \sum_k (c(n, k, i) - \alpha_k) x_k | < \sum_k | c(n, k, i) - \alpha_k | N^{1/p_k} .
\]
By (i) and (iii) we have
\[
\lim \sum_k c(n, k, i) x_k = \sum_k \alpha_k x_k
\]
uniformly in \( i. \) Hence proof is complete.

Proof of (b) is obvious if we take \( \alpha_k = 0. \)

**Theorem 3.3 (a)** \( A \in (\hat{c}(p), F_{\bar{B}}) \) if and only if

(i) conditions (i), (ii) and (iii) of Theorem (3.1) hold.

(ii) \( \lim \sum_n \sum_j b_{nj} \cdot (a_{jk} - a_{j,k+1}) - (\alpha_k - \alpha_{k+1}) |B|^{1/p_k} = 0 \)

(b) \( A \in (\hat{c}(p), F_{\bar{B}})_{\text{reg}} \) if and only if conditions (i), (ii) with \( \alpha_k = 0, \) (iii) with \( \alpha = 1 \) and (ii) of (a) hold.

**Proof (a)** Necessity. Let \( A \in (\hat{c}(p), F_{\bar{B}}). \) Now by cirture of the fact \( N(B_i) < \infty, A : c(p) \rightarrow F_{\bar{B}} \) and Theorem (3.1) follows all the conditions of (i). To prove condition (ii), let us define a matrix \( G = (g_{nk}) \) with
\[
g_{nk} = \begin{cases} 
1, & n = k, \\
-1, & n = k + 1, \\
0, & \text{otherwise},
\end{cases}
\]
and the matrix \( \bar{B}_k \) with \( \bar{B}_k = (b_{kj}^{(1)}(i)), 0 \leq i, j < \infty \) we see that it is easy to prove the following conditions:

(iii) \( G : l_\infty (p) \rightarrow \hat{c}_o (p) \)

(iv) \( G^{-1}(G \cdot x) = x \) with \( G^{-1} = (g_{nk}^{(-1)}) \)
\[
(1) = \begin{cases} 
1, & 0 \leq k \leq n, \\
0, & \text{otherwise}.
\end{cases}
\]
(v) \( N[\mathbf{G}^\top (I - \mathbf{B}_k)] = k.\)

Let us choose \( x \in l_\infty (p) \). Then by (iii), \( G x \in e_\circ (p) \) and 
\[
A(Gx) = D x \in F_B^D \quad \text{i.e.} \quad l_\infty (p) \longrightarrow F_B^D. \] 
Thus by

Theorem (3.2), condition (ii) follows immediately.

**Sufficiency.** Suppose (i) and (ii) holds and \( x \in \hat{e} (p) \). We have to show that \( A x \in F_B^D \). Since \( x \in \hat{e} (p) \) implies

\[
| t_{n,i} (x - 1 e) |^{p_k} \longrightarrow 0, \ n \longrightarrow \infty \text{ for some } l, \text{ uniformly in } i.
\]

Where

\[
t_{n,i} (x) = \frac{1}{n+1} \sum_{k=i}^{i+n} x_k.
\]

Hence for a given \( \varepsilon > 0 \) \( \exists k_0 \geq 0 \) such that \( \forall k < k_0 \)

\[
| t_{n,i} (x - 1 e) |^{p_k/M} < \frac{\varepsilon}{3 (\sum_k |x_k| + \sum_k |T_{in}(e_k)| + 1)}
\]

therefore \( B^{1/p_k} \) \( | t_{n,i} (x - 1 e) | \rightarrow B^{M/p_k} \) \( | t_{n,i} (x - 1 e) | \)

\[
< \frac{\varepsilon}{3 (\sum_k |x_k| + \sum_k |T_{in}(e_k)|)}
\]

where

\( T_{in} (x) = (B_i (A x))_n. \) Now we have

\[
T_{in} (x) = \sum_k (T_{in} (e_k)) x_k + (T_{in} (e) - \sum_k T_{in} (e_k)) (\hat{e} - \lim x).
\]

By given conditions, we have

\[
\lim_{n} T_{in} (e_k) = x_k \text{ and}
\]

\[
\lim_{n} T_{in} (e) = x \text{ uniformly in } i.
\]

And hence \( \exists n_0 \geq r \) with

\[
\sum_{k=0}^{k_0} |x_k - T_{in} (e_k)| < \frac{\varepsilon}{3 (2 |\hat{e} - \lim x| + 1)}
\]
\[ |x - T_{in}(e)| < \frac{\varepsilon}{3 (|\delta - \lim x| + 1)} \]

which is true for all \( i \geq 0 \) and \( n \geq n_0 \). Now by Banach–Steinhaus Theorem \( L_1 \in \delta ' (p) \) (continuous dual space of \( \delta (p) \)), where

\[
L_1 x = \sum_k (L_1 e_k) x_k + (L_1 e - L_1 e_k) (\delta - \lim x)
\]

\[
= \sum_k \alpha_k x_k + (\alpha - \sum_k \alpha_k) (\delta - \lim x)
\]

\[= L_1 x.\]

Therefore

\[
|L_1 x - T_{in}(x)| = |(x - T_{in}(e)) (\delta - \lim x) + \sum_k (\alpha_k - T_{in}(e_k)) (t_{n,i} (x - 1 e))|
\]

\[
\leq |x - T_{in}(e)||\delta - \lim x| + 2|\delta - \lim x| \sum_{k=0}^{k_0} |\alpha_k - T_{in}(e_k)|
\]

\[+ \sup_{k_0 < k < \infty} |t_{n,i} (x - 1 e)| \left( \sum_{k=0}^{k+1} |\alpha_k| + \sum_{k=0}^{k+1} |T_{in}(e_k)| \right)
\]

\[< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]

Hence \( A \in (\delta (p), F_{\mathcal{B}}) \).

Proof of (b) is immediate if we observe that \( \alpha_k = 0 \) and \( x = 1 \) in (a).

Finally the author is grateful to Dr. Z. U. Ahmad for his suggestions and guidance.

ÖZET

Bu çalışmada amacımız, hingga dek ortaya atılmış olan matris sınıflarındaki bir boşluğu dolduracak matrisler sınıflarını incelemektir. Daha önce \( (\mathcal{C}_0 \ (p), F_{\mathcal{B}}, l \ (p), F_{\mathcal{B}}, W \ (p), F_{\mathcal{B}}, W_{\mathcal{P}} \ (p), F_{\mathcal{B}} \) ve \( c \ (p), F_{\mathcal{B}}, c \ (p), F_{\mathcal{B}} \) matrislerinin karakterize etmiştir. Bu araştırmamızda \( (W \ (p), F_{\mathcal{B}}), (W_{\mathcal{P}} \ (p), F_{\mathcal{B}}), (c \ (p), F_{\mathcal{B}}), (l_{\infty} \ (p), F_{\mathcal{B}}, (c \ (p), F_{\mathcal{B}}) \) ve \( (c \ (p), F_{\mathcal{B}}) \) matrislerini karakterize edeceğiz.
REFERENCES


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