Left And Right Spectra

by

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Left And Right Spectra

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ABSTRACT.

The left spectrum $\sigma^l(a)$ and the right spectrum $\sigma^r(a)$ of an element in a Banach algebra $A$ are considered and some properties are proved. Operator algebras in which, for every element $T$, $\sigma^l(T) = \sigma^r(T)$ are investigated, and a characterization of $\sigma^l(T)$ and $\sigma^r(T)$ is given.

INTRODUCTION

The left spectrum $\sigma^l(a)$ and the right spectrum $\sigma^r(a)$ of an element $a$ in a Banach algebra $A$ with identity are defined to be the following subsets of the field $C$ of complex numbers:

$$\sigma^l(a) = \{\lambda \in C : a - \lambda e \text{ is not left invertible}\}$$

$$\sigma^r(a) = \{\lambda \in C : a - \lambda e \text{ is not right invertible}\}.$$  

Equivalently, $\lambda \in \sigma^l(a)$ (or $\sigma^r(a)$) if and only if $a - \lambda e$ generates a proper left (right) ideal in $A$. If the algebra $A$ is commutative then

$$\sigma^l(a) = \sigma^r(a) = \{\psi(a) : \psi \in \Phi\}$$

where $\Phi$ is the maximal ideal space of $A$ [1, p. 320]. For an element $a$ in a noncommutative algebra $A$, $\sigma^l(a) = \sigma^r(a)$ is not true in general.

The notion was first introduced by Robin Harte ([2] [3]) to prove spectral mapping theorems for the joint spectrum of an n-tuple $a = (a_1, a_2, \ldots, a_n)$ in $A$. In the present paper we shall prove some properties of $\sigma^l(a)$ and $\sigma^r(a)$, and we shall give a characterization of $\sigma^l(T)$ and $\sigma^r(T)$ for an element $T$ in the Banach algebra $A$ of operators on a Banach space.
II. PROPERTIES OF $\sigma(a)$ AND $\sigma'(a)$

Let $A$ be a Banach algebra with identity $e$, and $a \in A$. It is well known that $\sigma(a) = \sigma'(a)$. $\sigma'(a)$ is a non-empty compact subset of $C$ contained in the disk $\{z \in C : |z| \leq \|a\| \}$. Now we note that $\sigma'(a)$ or $\sigma'(a)$ can be proper subsets of $\sigma(a)$. This is demonstrated by the following example.

**Example.** Let $H = F$ and $A$ be the Banach algebra of all bounded linear operators on $H$. Then for any $T \in A$,

$$\sigma'(T) = \{ \lambda \in \mathbb{C} : \inf \{ \| (T - \lambda)x \| = 0 \}, \| x \| = 1 \}$$

$$\sigma''(T) = \{ \lambda \in \mathbb{C} : (T - \lambda) \neq H \}$$

[3, pp. 95-97]. Therefore if we take an operator $T \in A$ which is not one-to-one but onto, then $0 \in \sigma'(T)$ but $0 \notin \sigma''(T)$. For instance define $T$ by

$$T(x) = (x_1, x_2, x_3, \ldots) \text{ for } x = (x_1, x_2, x_3, \ldots).$$

It is easy to see that $T$ is linear, and bounded since

$$\| T(x) \|^2 = \sum_{n=1}^{\infty} |x_{2n-1}|^2 \leq \| x \|^2.$$ 

We observe that $T$ is onto. If $y = (y_1, y_2, y_3, \ldots)$ is in $H$, then $T(x) = y$ for $x = (y_1, 0, y_2, 0, y_3, \ldots)$. We note that $\ker T \neq \{0\}$, since $\ker T$ consists of all vectors $x$ of the form $x = (0, x_2, 0, x_4, 0, x_6, \ldots)$.

Since $\sigma'(a)$ or $\sigma''(a)$ could be proper subsets of $\sigma(a)$ it is natural to ask whether either of them can be empty. We shall prove that neither $\sigma'(a)$ nor $\sigma''(a)$ can be empty.

An element $a$ in $A$ is said to be a left (right) topological zero divisor if there exists a sequence $\{b_n\}$ in $A$ such that $\|b_n\| = 1$, $n = 1, 2, 3, \ldots$, and

$$\lim_{n \to \infty} \|ab_n\| = 0 \quad (\lim_{n \to \infty} \|b_na\| = 0),$$

and $a$ is said to be a two-sided topological zero divisor if there exists a sequence $\{b_n\}$ in $A$ for which $\|b_n\| = 1$, $n = 1, 2, 3, \ldots$, and

$$\lim_{n \to \infty} \|ab_n\| = 0 = \lim_{n \to \infty} \|b_na\|.$$
**Theorem 1.** \( \sigma'(a) \) and \( \sigma^r(a) \) are both non-void compact subsets of \( C \). Furthermore the boundary of \( \sigma(a) \) (bdy \( \sigma(a) \) ) is included in both \( \sigma'(a) \) and \( \sigma^r(a) \).

**Proof.** We give the proof for the left spectrum. The proof for the right spectrum is similar. Let \( \lambda \in \text{bdy} \ \sigma(a) \). Then \( a - \lambda \ e \) is a boundary point of the group \( G \) of regular elements, therefore \( a - \lambda \ e \) is a two-sided topological zero divisor [4, p. 862]. We claim that \( \lambda \in \sigma'(a) \). If \( b \in A \) is a left inverse for \( a - \lambda \ e \), then \( b \ (a - \lambda \ e) = e \) implies that \( b_n = b \ (a - \lambda \ e) \ b_n \) and hence there is inequality

\[
\| b_n \| \leq \| b \| \| (a - \lambda \ e) \ b_n \|
\]

which rules out the possibility that \( a - \lambda \ e \) is a left topological zero divisor. So, \( \lambda \in \sigma'(a) \). Similarly, \( a - \lambda \ e \) is a right topological zero divisor implies that \( \lambda \in \sigma^r(a) \), and the proof is complete.

**Definition.** A complex linear algebra \( A \) with identity \( e \) will be called semi-commutative if \( \sigma'(a) = \sigma^r(a) \) for every element \( a \) in \( A \).

Of course every commutative algebra is semi-commutative. It is interesting to investigate semi-commutative algebras which are not commutative. An example of such an algebra which comes first to the mind is the algebra \( A \) of \( n \times n \) complex matrices. If \( a \in A \) then \( \lambda \in \sigma'(a) \) if and only if \( a - \lambda \ e \) is not left invertible but a square matrix is left invertible in and only if it is right invertible. Therefore, \( \sigma'(a) = \sigma^r(a) = \sigma(a) \). In this case \( \sigma(a) \) is the set of eigenvalues of the \( n \times n \) order complex matrix \( a \).

A semi-commutative algebra can easily be characterized as follows:

**Proposition.** A Banach algebra \( A \) with identity \( e \) is semi-commutative if and only if for any two elements \( a, b \) in \( A \)

\[ ab = e \text{ if and only if } ba = e \]

that is, an element \( a \) is left invertible if and only if it is right invertible.

In a Banach algebra \( A \) it is possible to have \( ab = e \neq ba \). For example, let \( A \) be the Banach algebra of all bounded linear operators on the Hilbert space \( l^2 \). Consider the right and left shifts \( S_R \) and \( S_L \) defined by
\[ S_R (x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots), \]
\[ S_L (x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots), \]

If is easy to see that \( S_L \cdot S_R = 1 \neq S_R \cdot S_L \). Of course this algebra cannot be semi-commutative according to our preceding proposition, for instance one can show that \( \sigma'(S_R) \neq \sigma'(S_R) \). We note that \( \sigma'(S_R) = \{0\} \), but \( 0 \notin \sigma'(S_R) \). To see this we recall that \( \lambda \in \sigma'(S_R) \) if and only if \( S_R - \lambda I \) is not onto. But \( S_R - \lambda I \) is onto for any \( \lambda \neq 0 \), since if \( y = (y_1, y_2, y_3, \ldots) \) is in \( V \) then \( (S_R - \lambda I)(x) = y \) for \( x = (x_1, x_2, x_3, \ldots) \) where \( x_1 = \frac{y_1}{\lambda}, \ x_2 = \frac{x_1 - y_2}{\lambda}, \)

\[ x_3 = \frac{x_2 - y_3}{\lambda}, \ldots, \ x_n = \frac{x_{n-1} - y_n}{\lambda}, \] for any \( n = 2, 3, 4, \ldots \). Hence \( \sigma'(S_R) = \{0\} \). Now we show that \( 0 \notin \sigma'(S_R) \). Again we recall that \( \lambda \in \sigma'(S_R) \) if and only if \( \inf \| (S_R - \lambda I)(x) \| = 0 \).

\[ \| x \| = 1 \]

But \( \| S_R(x) \|^2 = \sum_{i=1}^{\infty} |x_i|^2 = \| x \|^2 \). Therefore \( \inf \| S_R(x) \| = 1, \)

\[ \| x \| = 1 \]

and hence \( 0 \notin \sigma'(S_R) \).

**Theorem 2.** Every finite dimensional Banach algebra with identity is semi-commutative.

**Proof.** Let \( A \) be a Banach algebra with identity \( e \), and let \( L (A) \) be the Banach algebra of all bounded linear operators on \( A \). We identify \( A \) with the subalgebra of \( L (A) \) consisting of the operators \( T_a, \ a \in A \), where \( T_a(b) = ab \) If the dimension of \( A \) is \( n \), then \( L (A) \) is isomorphic to \( C^{nxn} \), \( nxn \) matrices. Therefore \( A \) is isomorphic to an \( n \)-dimensional subspace of \( C^{nxn} \). Let \( e_1, e_2, \ldots, e_n \) be the the standard basis of \( C^n \) and \( M_a \) be the matrix of \( T_a \) relative to this basis. Then for any \( a \in A \) we have

\[ \sigma_a (a) = \sigma_{L(A)} (T_a) = \sigma_{C^{nxn}} (M_a) \]

where \( \sigma \) denotes the spectrum of any sort left or right. But we have already observed that for any \( n \)th order complex matrix \( M_a \), \( \sigma'(M_a) = \sigma'(M_a) \). Therefore for any \( a \in A \) we have \( \sigma'(a) = \sigma'(a) \), and \( A \) is a semi-commutative algebra.
In our previous discussions, we proved that in a non-commutative Banach algebra \( A \) it is not always true that \( \sigma^f(a) = \sigma^r(a) \) for every \( a \in A \). It is interesting to know for which elements \( \sigma^f(a) = \sigma^r(a) \), in case of an algebra whose structure is familiar to us. We shall answer this question in case of the Banach algebra of all bounded linear operators on a Hilbert space \( H \). We know that in an algebra of linear operators on a finite dimensional space, it is always true that \( \sigma^f(T) = \sigma^r(T) \) for every operator \( T \). Many of the results that hold for linear transformations on finite dimensional space also hold in the infinite-dimensional case, provided the additional hypothesis of compactness is imposed.

**Theorem 3.** Let \( H \) be a Hilbert space, and \( A = \mathcal{L}(H) \) be the Banach algebra of all bounded linear operators on \( H \). If \( T \) is a compact operator and \( \lambda \neq 0 \) is a complex number, then \( \lambda \in \sigma^f_A(T) \) if and only if \( \lambda \in \sigma^r_A(T) \).

**Proof.** We recall once more that for any \( T \in A \) we have

\[
\sigma^f(T) = \{ \lambda \in C : \inf_{\|x\| = 1} \| (T - \lambda I) (x) \| = 0 \} \\
\sigma^r(T) = \{ \lambda \in C : (T - \lambda I) H \neq H \}.
\]

If \( T \) is a compact operator and \( \lambda \notin \sigma^f(T) \) for \( \lambda \neq 0 \), then \( \inf_{\|x\| = 1} \| (T - \lambda I) (x) \| > 0 \), i.e., \( T - \lambda I \) is one-to-one. But this is true if and only if \( T - \lambda I \) is onto [5, pp. 393-393]. So \( \lambda \notin \sigma^f(T) \). Similarly, if \( \lambda \notin \sigma^r(T) \) then \( (T - \lambda I) H = H \), i.e., \( T - \lambda I \) is onto. But this is true if and only if \( T - \lambda I \) is one to one. Thus, clearly \( \inf_{\|x\| = 1} \| (T - \lambda I) (x) \| > 0 \), and \( \lambda \notin \sigma^f(T) \).

We can not sharpen the statement of theorem 3 to conclude that \( \sigma^f(T) = \sigma^r(T) \) for every compact operator \( T \) in \( A = \mathcal{L}(H) \). The point \( \lambda = 0 \) has a status different from other points in relation to \( T \) if \( T \) is compact and \( H \) is infinite dimensional. In this case 0 is always in the spectrum \( \sigma(T) = \sigma^f(T) \cup \sigma^r(T) \), because the Banach subalgebra of all compact operators in \( A \) is a two-sided ideal in \( A \) which is not inverse closed [6, pp. 98-991].
Corollary 1. Let $A$ be the Banach algebra of all bounded linear operators on a Hilbert space $H$. Then $\sigma'(T) = \sigma'(T)$ for every finite rank operator $T$.

Proof. If $H$ is finite dimensional then $\sigma'(T) = \sigma'(T)$ for every $T$. Suppose that $H$ is infinite dimensional. If $T$ is a finite rank operator then it is compact, and furthermore $0 \in \sigma'(T) \cap \sigma'(T)$ because a finite rank operator can never be one-to-one, and it can never be onto if $H$ is infinite dimensional. If $\lambda \neq 0$, then by theorem 3, $\lambda \in \sigma'(T)$ if and only if $\lambda \in \sigma'(T)$, and the proof is complete.

Corollary 2. Let $A$ be the Banach algebra of all bounded linear operators on a Hilbert space $H$, and let $T$ be a compact operator. Then every $\lambda \neq 0$ in $\sigma(T)$ is an eigenvalue of $T$.

Proof. If $\lambda \neq 0$, $\lambda \in \sigma(T)$ then by theorem 3 $\lambda$ is necessarily in $\sigma'(T)$, therefore $\inf \| (T-\lambda \ I) (x) \| = 0$. Thus, $T-\lambda \ I \quad \|x\|=1$

is not one-to-one, and $\lambda$ is an eigenvalue of $T$.

Although $0$ is always in $\sigma(T)$ for a compact operator $T$, $0$ need not be an eigenvalue of $T$.

Example. Let $H=L^2$, and let $e_1 = (1, 0, 0, ...)$, $e_2 = (0, 1, 0, ...)$, $e_3 = (0, 0, 1, 0, ...)$ be the standard complete orthonormal set in $H$. For $x = (x_1, x_2, x_3, ...) \in H$ we define an operator $T$ by

\[ T \ (x) = (0, \ \frac{x_1}{2}, \ \frac{x_2}{3}, \ \frac{x_3}{4}, \ ...). \]

We show that $T$ is a compact operator. If we define the sequence of operators $\{T_n\}$ by

\[ T_n \ (x) = (0, \ \frac{x_1}{2}, \ \frac{x_2}{3}, \ ... , \ \frac{x_n}{n+1}, \ 0, \ 0, ...) \]

for $n=1,2,3, ...$ then it is a Cauchy sequence in the norm topology of $L(H)$, and therefore convergent. Clearly, $\lim_{n \to \infty} T_n = T$. Each $T_n$ being a finite rank operator is compact and therefore, $\lim_{n \to \infty} T_n = T$ is compact because the Banach subalgebra of all compact operators is the norm closure of the finite rank operators [7, pp. 124-125].

If $T \ (x) = 0$, then obviously $x$ must be zero, therefore $T$ is one-to-one. Thus $0$ is not in $\sigma'(T)$ but $0 \in \sigma'(T)$ since $T$ is not onto.
III. A CHARACTERIZATION OF \( \sigma^l (T) \) and \( \sigma^r (T) \)

Let \( X \) be a Banach space and \( A = L (X) \) be the Babach algebra of all bounded linear operators on \( X \). We shall denote the set of all left (right) invertible elements in \( A \) by \( G^l(G^r) \). We set \( G = G^l \cap G^r \). We note that \( T \in G \) if and only if \( T \) is a topological isomorphism (i.e., a linear isomorphism which is also a homeomorphism) onto \( X \).

**Theorem 4.** \( T \in G^l \) if and only if \( T \) is a topological isomorphism between \( X \) and the range of \( T \), and there is a projection of \( X \) on the range of \( T \).

**Proof.** If \( T \in G^l \) then \( T \) is not a left topological zero divisor and this implies that \( T \) is a topological isomorphism between \( X \) and the range of \( T \). To prove the existence of a projection of \( X \) on the range of \( T \) we first show that \( \text{ran} \ T \) is a closed subspace. Since \( T \) is a topological isomorphism, \( T \) is bounded below, i.e., there exists an \( \varepsilon > 0 \) such that \( \| T (x) \| > \varepsilon \| x \| \) for every \( x \) in \( X \). Hence, if \( \{ T(x_n) \}_{n=1}^\infty \) is a Cauchy sequence in \( \text{ran} \ T \), then the inequality

\[
\| x_n - x_m \| < \frac{1}{\varepsilon} \| T(x_n) - T(x_m) \|
\]

implies that \( \{ x_n \} \) is also a Cauchy sequence. If \( x = \lim x_n \), then \( T(x) = \lim T(x_n) \) is in \( \text{ran} \ T \). Thus \( \text{ran} \ T \) is closed.

Let \( S \) be the inverse mapping from \( Y = \text{ran} \ T \) to \( X \). Then \( ST = I \) in \( A \). By hypothesis there exists \( U \) in \( A \) such that \( UT = I \) in \( A \). Consequently \( U = S \) on \( Y \) and \( U \) is an extension of \( S \). Now we decompose \( X \) into cosets \( y + \ker U \), \( y \in Y \). By hypothesis each coset \( y + \ker U \) contains one and only one \( y \in Y \), and every element of \( X \) is included in some coset since \( U \) is defined on all of \( X \). Thus each \( x \in X \) has a unique decomposition \( x = y + (x-y) \) where \( y \in Y \) is the representative of the coset to which \( x \) belongs, so that \( x-y \in \ker U \). Therefore \( Y \) and \( \ker U \) are complementary subspaces in \( X \), and the transformation defined by \( P(x) = y \) is a projection on \( X \) to \( Y = \text{ran} \ T \). Since both the range and the kernel of \( P \) are closed, \( P \) is bounded [8, p. 242].
Conversely let $T$ be a topological isomorphism between $X$ and the range of $T$, and suppose that a bounded projection $P$ of $X$ on ran $T$ exists. Let $S$ be the inverse mapping between ran $T$ and $X$. Then $SP$ is a bounded operator with domain all of $X$. Furthermore $(SP)T = I$ and thus $T \in G^l$

**Corollary 1.** If $T$ is an operator on a Hilbert space $H$ then $T \in G^l$ if and only if $T$ is bounded below.

**Proof.** $T$ is bounded below if and only if $T$ is an isomorphism between $H$ and the closed subspace ran $T$. Since $H$ is a Hilbert space there exists a projection of $H$ onto the closed linear subspace ran $T$ and the corollary follows from theorem 4.

**Corollary 2.** If $T$ is an operator on the Hilbert space $H$, then $\lambda \in \sigma^l(T)$ if and only if $\inf \| (T-\lambda I)(x) \| = 0\| x \| = 1$

This is a restatement of Corollary 1 in terms of left spectrum.

**Theorem 5.** $T \in G^c$ if and only if $T$ is onto and there exists a projection of $X$ onto Ker $T$.

**Proof.** Suppose $T \in G^c$. Then $T$ is not a right topological zero divisor. We know that ran $T = X$ if $T'$ is a topological isomorphism [8, p. 234]. Assume the contrary that $T'$ is not an isomorphism. Then there exists a sequence $\{x_n\} \subset X'$ with $\| x_n \| = 1$ such that
\[
\lim_{n \to \infty} \| T'(x_n) \| = 0, \text{ or } \lim_{n \to \infty} | x_n'(T_n) | = 0 \text{ for every } x \text{ in the closed unit ball of } X.
\]
Let $u \in X, \| u \| = 1$; and let $U_n \in A$ be defined by $U_n(x) = x_n'(x) u$ for $n = 1, 2, 3, \ldots$. It is easy to show that $\| U_n \| = 1$, and also $\lim_{n \to \infty} \| U_n(T_n) \| = \lim_{n \to \infty} | x_n'(T_n) u | = 0$ for every $x$ with $\| x \| \leq 1$, which contradicts the fact that $T$ is not a right topological zero divisor. Consequently ran $T = X$.

To prove the existence of a projection of $X$ on Ker $T$ we show that $X$ is the direct sum $X = \text{Ker } T \oplus \text{ran } U$ where $U$ is a right inverse for $T$, i.e. $TU = I$. Ker $T \cap \text{ran } U = \{0\}$, for if $U(x) \neq 0$ and $U(x) \in \text{Ker } T$ then $TU = I$ is violated.
We consider the quotient space $X/\ker T$, and show that every coset $x + \ker T$ contains one and only one element of $\text{ran} \ U$. Suppose that $x_0 + \ker T$ contains two elements $y_1$ and $y_2$ of $\text{ran} \ U$. Let $y_1 = U(x_1)$ and $y_2 = U(x_2)$. Since $y_1 - y_2 \in \ker T$ we have $TU(x_1) = TU(x_2)$ or $x_1 = x_2$, and hence $y_1 = y_2$. On the other hand $x_0 + \ker T$ contains an element of $\text{ran} \ U$. For every $x \in x_0 + \ker T$, $T(x)$ has the same value $T(x_0)$, moreover $T(x) = T(x_0)$ only if $x \in x_0 + \ker T$. Now we note that $TU(x_0) = T(x_0)$. Then $z = UT(x_0)$ is in $(x_0 + \ker T) \cap \text{ran} \ U$. Let $x \in X$, and let $Y$ be a coset of $X/\ker T$ which contains $x$. Let $x_1$ be the unique representative of $Y$ in $\text{ran} \ U$. Then $x$ has the representation $x = x_1 + (x-x_1)$ where $x_1 \in \text{ran} \ U$ and $x-x_1 \in \ker T$ (since both $x$ and $x_1$ are in $Y$). This representation is unique. For if also $x = x_2 + (x-x_2)$ where $x_2 \in \text{ran} \ U$ and $x_2 \neq x_1$ then $x_2 \notin Y$, because $Y$ contains exactly one element of $\text{ran} \ U$. Since $x \in Y$, $x-x_1$ is not in $\ker T$. Consequently $x = x_1 + \ker T \cap \text{ran} \ U$. Since $TU = I$, $U \in G'$ and by Theorem 4 $U$ is a topological isomorphism, and thus $\text{ran} \ U$ is closed. Therefore $\ker T$ and $\text{ran} \ U$ are closed complementary subspaces, and there exists a bounded projection of $X$ on $\ker T$ [8, p. 242].

Conversely, suppose that $T$ is onto and there exists a bounded projection $P_i$ of $X$ on $\ker T$. Then $X = \text{ran} \ P_i \oplus \ker P_i = \ker T \oplus \ker P_i$. If we let $P = I - P_i$, then $\text{ran} \ P = \ker P_i$ and $X = \ker T \oplus \text{ran} \ P$. If we consider $TP$ as a mapping with domain $\text{ran} \ P$ and range in $X$, then $TP$ is a topological isomorphism between $\text{ran} \ P$ and all of $X$. Let $x_1$ and $x_2$ be in $\text{ran} \ P$. Then $P(x_1) = x_1$ and $P(x_2) = x_2$. If $TP(x_1) = TP(x_2)$, then $T(x_1 - x_2) = 0$ and $x_1 - x_2 \in \ker T \cap \text{ran} \ P = \{0\}$. Thus $TP$ is a one-to-one mapping. To see that the range of $TP$ is all of $X$, take any $y \in X$. Since $\text{ran} \ T = X$, there exists an element $x \in X$ such that $T(x) = y$. Let $x = x_1 + x_i$ be the decomposition of $x$ where $x_1 \in \ker T$ and $x_i \in \text{ran} \ P$. Then $y = T(x_1) + T(x_i) = T(x_1) = TP(x_2)$. Then by the Open Mapping Theorem $TP$ is a topological isomorphism. Let $S$ be the inverse mapping from $X$ to $\text{ran} \ P$. Then $(TP) S = I = T (PS)$ and $PS \in \mathbb{A} = L(X)$, consequently $T \in G'$.  

**Corollary 1.** If $T$ is an operator on a Hilbert space $H$ then $T \in G'$ if and only if $T$ is onto.
Proof, By Theorem 5, \( T \in G \) if and only if \( T \) is onto and there exists a projection of \( H \) on \( \text{Ker} \, T \). Since \( H \) is a Hilbert space there always exists a projection on the closed linear subspace \( \text{Ker} \, T \).

Corollary 2. If \( T \) is an operator on the Hilbert space \( H \) then \( \lambda \in \sigma^r(T) \) if and only if \( T - \lambda I \) is not onto.

This is a restatement of Corollary 1 in terms of the right spectrum of \( T \).

REFERENCES


ÖZET

Sol ve Sağ Spektrumlar

Bir \( A \) Banach cebiri içindeki bir \( a \) elemanının \( \sigma^l(a) \) sol spektrumu ve \( \sigma^r(a) \) sağ spektrumu incelenmektede ve bazı özellikleri ispatlanmaktadır. Her \( T \) eleman için \( \sigma^l(T) = \sigma^r(T) \) olan operatör cebirleri araştırılmakta ve \( \sigma^l(T) \) ve \( \sigma^r(T) \) cümlelerinin bir karekterizasyonu verilmektedir.
Prix de l'abonnement annuel

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