ON THE 3–PLANE AND CONE AND SPHERES

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SUMMARY

Elkholy and Areefi showed that in a space time, the intersection of a plane, passing through the origin, with the light cone, given by the equation \( \sum_{i=1}^{3} x_i^2 - x_4^2 = 0 \), is two 2–planes perpendicular to each other. In this study, instead of Elkholy-Areefi’s light cone in a space time by dealing with the cone given by the equation, \( a'_1 x_1^2 + a'_2 x_2^2 + a'_3 x_3^2 - b x_4^2 = 0 \) and showing that also its intersection with 3–plane, passing through the origin, is two 2–planes perpendicular to one another, the generalization of the article of Elkholy-Areefi has been obtained. Furthermore, validity is proved for the sphere given by the equation

\( \sum_{i=1}^{3} x_i^2 + z = x_4^2 \).

I. Introduction

1.2. Definition

A diametral plane is known by the equation

\[
\frac{l}{\partial F} + m \frac{\partial F}{\partial y} + n \frac{\partial F}{\partial z} = 0
\]  

(1)

where

\[ F(x,y,z)=ax+by+cz+2fyz+2gzx+2hxy+d=0 \] [1].

Calculating \( \frac{\partial F}{\partial x} \), \( \frac{\partial F}{\partial y} \) and \( \frac{\partial F}{\partial z} \), the equation of diametral plane is obtained as

\[ x(al+hm+gn)+y(hl+bm+fn)+z(gl+fm+cn)=0. \]  

(2)
1.2. Definition:

If the normal of a diametral plane is linearly dependent to the vector \((l, m, n)\), then the diametral plane given by (2) is called perpendicular to the line

\[
\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.
\]

If the diametral plane is perpendicular to the line

\[
\frac{x}{l} = \frac{y}{m} = \frac{z}{n}
\]
then the homogeneous system of linear equations,

\[
\begin{align*}
(a-\lambda)l+hm+ gn &= 0 \\
h(l+(b-\lambda)m+fn &= 0 \\
gl+fm+(c-\lambda)n &= 0
\end{align*}
\]

is obtained. To have non-trivial solutions, the coefficient determinant must be zero for this equation system. That is,

\[
\lambda^3-\lambda^2(a+b+c)+\lambda(bc+ca+ab-h^2-g^2-f^2)-D=0
\]

where,

\[
D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}
\]

1.3. Definition

The equation,

\[
\lambda^3-\lambda^2(a+b+c)+\lambda(bc+ca+ab-h^2-g^2-f^2)-D=0
\]
is called the cubic discriminating of \(F(x, y, z)\) \([1]\).

Regarding to the equation in (4), for \(\lambda\) there are at most three solutions. For each \(\lambda_i, 1 \leq i \leq 3\), we can find the three non-trivial solutions \((l_i, m_i, n_i)\). So,

\[
l_ix+m_iy+n_iz=0, \quad 1 \leq i \leq 3,
\]
diametral planes are obtained.
II. The Main Results

Let

\[ A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 = 0 \]

be a 3-plane passing through the origin. Getting \( \frac{A_i}{A_4} = B_i \), \( 1 \leq i \leq 3 \), the equation of this plane becomes,

\[ \sum_{i=1}^{3} B_i x_i + x_4 = 0. \]  \hspace{1cm} (5)

On the other hand, for the cone given by

\[ \sum_{i=1}^{3} a_i' x_i^2 - b x_4^2 = 0 \]

substituting \( \frac{a_i'}{b} = a_i \), the equation reduces to,

\[ \sum_{i=1}^{3} a_i x_i^2 - x_4^2 = 0. \]  \hspace{1cm} (6)

From (5) and (6) we have that

\[ \left( \sum_{i=1}^{3} B_i x_i \right)^2 = \sum_{i=1}^{3} a_i x_i^2 \]

or

\[ \Rightarrow \sum_{i=1}^{3} (B_i - a_i) x_i^2 + \sum_{i,j=1}^{3} B_i B_j x_i x_j = 0, \]

If we denote \( B_i - a_i = C_j \), we have the quadric

\[ F(x_1, x_2, x_3) = C_1^2 x_1^2 + C_2^2 x_2^2 + C_3^2 x_3^2 + 2B_1 B_2 x_1 x_2 + 2B_1 B_3 x_1 x_3 + 2B_2 B_3 x_2 x_3 = 0. \] \hspace{1cm} (7)

The diametral plane of this quadric can be given as,

\[ l \frac{\partial F}{\partial x_1} + m \frac{\partial F}{\partial x_2} + n \frac{\partial F}{\partial x_3} = 0. \] \hspace{1cm} (8)
Calculating \( \frac{\partial F}{\partial x_i} \), \( 1 \leq i \leq 3 \), we can have

\[
(C_1 l + B_1 B_2 m + B_1 B_3 n) x_1 + (B_1 B_2 l + C_2 m + B_2 B_3 n) x_2 +
(B_1 B_2 l + B_2 B_3 m + C_3 n) = 0.
\]  
(9)

If we consider that this diametral plane is perpendicular to the line

\[
\frac{x_1}{l} = \frac{x_2}{m} = \frac{x_3}{n}
\]

then we have,

\[
\frac{C_1 l + B_1 B_2 m + B_1 B_3 n}{l} = \frac{B_1 B_2 l + C_2 m + B_2 B_3 n}{m} = \frac{B_1 B_2 l + B_2 B_3 m + C_3 n}{n} = \lambda,
\]

and therefore we can write the homogeneous system of linear equations,

\[
\left\{
\begin{aligned}
(C_1^2 - \lambda) l + B_1 B_2 m + B_1 B_3 n &= 0 \\
B_1 B_2 l + (C_2^2 - \lambda) m + B_2 B_3 n &= 0 \\
B_1 B_2 l + B_2 B_3 m + (C_3^2 - \lambda) n &= 0
\end{aligned}
\right\}.
\]  
(10)

The cubic discriminating of the equation (7) is

\[
\lambda^3 - \lambda^2(C_1^2 + C_2^2 + C_3^2) + \lambda(C_2^2 C_3^2 + C_1^2 C_2^2 + C_1^2 C_3^2) -
B_2^2 B_3^2 - B_2^2 B_3^2 - B_1^2 B_2^2) - D = 0
\]  
(11)

where

\[
D = \begin{vmatrix}
C_1^2 & B_1 B_2 & B_1 B_3 \\
B_1 B_2 & C_2^2 & B_2 B_3 \\
B_1 B_3 & B_2 B_3 & C_3^2
\end{vmatrix}
\]

Substituting \( B_1^2 = C_1^2 + a_1 \), equation (11) becomes,

\[
\lambda_3 - \lambda^2(C_1^2 + C_2^2 + C_3^2) - \lambda[(a_2 + a_3) C_1^2 + (a_1 + a_2) C_2^2 + a_1 a_2 + a_1 a_3 + a_2 a_3 + (a_1 + a_2) C_3^2] - [a_2 a_3 C_1^2 + a_1 a_2 C_2^2 + a_1 a_2 C_3^2 + 2a_1 a_2 a_3] = 0.
\]  
(12)

As a special case if we take \( a_1 = a_2 = a_3 = a \) in (6) the equation (12) becomes

\[
\lambda^3 - \lambda^2(C_1^2 + C_2^2 + C_3^2) - \lambda[2a(C_1^2 + C_2^2 + C_3^2) + 3a^2] - [a^2(C_1^2 + C_2^2 + C_3^2 + 2a^3)] = 0.
\]
For the sake of shortness, if we denote, \( A = C_1^2 + C_2^2 + C_3^2 \), then the cubic equation becomes,

\[
\lambda^3 - \lambda^2 A - \lambda [2a \cdot A + 3a^2] - [a^2 \cdot A + 2a^3] = 0 
\]

(13)

or

\[
(\lambda - a)^2 [\lambda - (A + 2a)] = 0.
\]

It follows that

\[
\lambda_1 = \lambda_2 = -a, \text{ and } \lambda_3 = A + 2a.
\]

Using \( \lambda_1 = \lambda_2 = -a \) in equation (10) we have

\[
B_1 l_1 + B_2 m_1 + B_3 n_1 = 0.
\]

(14)

And using \( \lambda_3 = A + 2a \) in (10) we have,

\[
\begin{align*}
-(B_1^2 + B_2^2) l_1 + B_1 B_2 m_3 + B_1 B_3 n_3 &= 0 \\
B_1 B_3 l_3 - (B_1^2 + B_2^2) m_3 + B_2 B_3 n_3 &= 0 \\
B_1 B_3 l_3 + B_2 B_3 m_3 - (B_1^2 + B_2^2) n_3 &= 0
\end{align*}
\]

(15)

Dividing the first equation of (15) by \( B_1 \) and the second by \( B_2 \) then subtracting, we have

\[
\frac{l_3}{B_1} = \frac{m_3}{B_2}.
\]

(16)

Again from the second and third equations, we can have

\[
\frac{m_3}{B_2} = \frac{n_3}{B_3}
\]

(17)

and then from (16) and (17)

\[
\frac{l_3}{B_1} = \frac{m_3}{B_2} = \frac{n_3}{B_3} = k, \quad k \in \mathbb{R}
\]

(18)

On the other hand, from solution (10) we have the diametral planes as

\[
\begin{align*}
l_1 x_1 + m_1 x_2 + n_1 x_3 &= 0 \\
l_3 x_1 + m_3 x_2 + n_3 x_3 &= 0
\end{align*}
\]

(19)

and from (18), these equations reduces to

\[
\begin{align*}
l_1 x_1 + m_1 x_2 + n_1 x_3 &= 0 \\
B_1 x_1 + B_2 x_2 + B_3 x_3 &= 0
\end{align*}
\]
where
\[ <(l_i, m_i, n_i), (B_1, B_2, B_3)> = l_i B_1 + m_i B_2 + n_i B_3 \]  
(20)

and from (14) it vanishes. So the planes given by (19) are perpendicular to each other.

As a result, we can write the following theorem:

II.1. Theorem:

The intersection of the cone \( \sum_{i=1}^{3} a'_i x_i^2 = bx_i^2 \) and the 3-plane \( \sum_{i=1}^{4} A_i x_i = 0 \) is two 2-planes, perpendicular to each other, if \( a'_1 = a'_2 = a'_3 \).

By using the Sz sphere \( \sum_{i=1}^{3} x_i^2 + \alpha = x_4^2 \) instead of the cone (6) all of the results are valid. So we can express the following theorem:

II.2. Theorem:

In \( (3+1) \)-spacetime, the intersection of the sphere \( \sum_{i=1}^{3} x_i^2 + \alpha = x_4^2 \) and the hyperplane \( \sum_{i=1}^{4} A_i x_i = 0 \) is two 2-planes, which are perpendicular to each other.

REFERENCES
