ON THE LINE CLASSES IN SOME FINITE HYPERBOLIC PLANES

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ABSTRACT

Determination of the line classes (and the number of lines in each class) in some hyperbolic planes of type $\pi_m$ occurs as an open problem. In this paper we give a partial answer to the problem for the special hyperbolic planes $\pi_2$, $\pi_4$, $\pi_6$, $\pi_7$, and $\pi_9^{n-2}$, $\pi_9^{n-1}$.

INTRODUCTION

It is well known that if a line is deleted from a projective plane then the remaining substructure forms an affine plane. Graves [1962], Ostrom [1962] and Bumcrot [1971] have given examples of hyperbolic planes obtained by deletion from projective planes. Graves [1962] also asked for additional constructions of such planes. Sandler [1963] has shown that if three non-concurrent lines are deleted from a projective plane then the remaining incidence structure forms a hyperbolic plane in the sense of Graves [1962]. Kaya-Özcan [1984] has extended the Sandler’s construction as follows: Let $\pi$ be a finite projective plane of order $n$ and $m$ a positive integer provided that $m < n + 2$. Let $\iota_1, \iota_2, \ldots, \iota_m$ denote distinct $m$ lines of $\pi$ such that no three are concurrent. Let $\pi_m$ be the substructure obtained by deleting from $\pi$ all of the lines $\iota_i$, $i = 1, 2, \ldots, m$, and all points on these $m$ lines. A point of $\pi$ is called corner point if it is intersection of any two lines in the set $\{\iota_1, \iota_2, \ldots, \iota_m\}$. Let $r$ denote the minimum number of corner points on a line of $\pi_m$ as a line of $\pi$. In [Kaya, R.-Özcan, E., (1984)] it has been shown that if $3 \leq m \leq n + r + \frac{1}{2}$ (1 – $\sqrt{4n + 5}$) then $\pi_m$ is a hyperbolic plane.

The lines of $\pi_m$ are classified according to the number of points which are contained in each line of a class. Let $C_s$ denote the set of all
lines of \( \pi_m \) such that each line in it contains exactly \( s \) corner points in \( \pi \). Each line in \( C_s \) contains exactly \( n+1-(m-s) \) points. There exist \( \tfrac{1}{2} \) (m-1)-r classes of lines in \( \pi_m \) according as \( m \) is an even or odd positive integer, respectively. The line classes are \( C_r \), \( C_{r+1}, \ldots, C_{m/2} \) or \( C_r,C_{r+1}, \ldots, C_{(m-1)/2} \) according as \( m \) is even or odd, respectively. It follows that if \( m \) is even then there exist exactly \( \tfrac{1}{2}m-r+1 \) classes of lines in \( \pi_m \) and \( \pi_{m+1} \) obtained from a projective plane, namely \( C_r,C_{r+1}, \ldots, C_{m/2} \). Furthermore, if \( q_s \) denote the number of all lines in \( C_s \) then one has the following:

\[
\begin{align*}
(I) \quad \sum_{s=r}^{t} q_s &= n^2+n+1-m \\
(II) \quad \sum_{s=r}^{t} sq_s &= (n-1) \binom{m}{2} \\
(III) \quad \sum_{s=r}^{t} s^2q_s &= \left[n-1+\binom{m-2}{2}\right] \binom{m}{2}
\end{align*}
\]

Where \( t \) is \( \frac{m}{2} \) or \( \frac{1}{2} (m-1) \) according as \( m \) is even or odd, respectively. (In what follows \( t \) will be used in that sense). One of the unsolved problems related to these hyperbolic planes is to determine the number of lines in each class \( C_s \) of \( \pi_m \). A partial answer to the problem is given in [Olgun, (1986)]. In the first part of this paper, we formulate the answer to the question for any finite planes of type \( \pi_3, \pi_4, \pi_5 \), and determine the required numbers for a \( \pi_6 \) and \( \pi_7 \) in terms of the number of lines in \( C_s \). In the second part, the problem is solved for some special hyperbolic planes of type \( \pi_{n-1} \) and \( \pi_{n-2} \). It would be very interesting to find the full answer to the above problem for any \( m \) and \( n \).

THE LINE CLASSES IN \( \pi_3, \pi_4, \pi_5, \pi_6, \) AND \( \pi_7 \).

PROPOSITION 1. For any hyperbolic plane \( \pi_m \)

\[
\begin{align*}
(i) \quad q_2 &= \frac{1}{2} \left( \binom{m}{2} \right) \left( \binom{m-2}{2} \right) - \sum_{s=3}^{t} \left( \binom{s}{2} \right) q_s \\
(ii) \quad q_1 &= \left( \binom{m}{2} \right) \left[ n-1+\binom{m-2}{2}\right] + \sum_{s=3}^{t} s(s-2)q_s
\end{align*}
\]
(iii) \( q_0 = n^2 + n \left[ 1 - \left( \frac{m}{2} \right) \right] + \left( \frac{m-1}{2} \right) + \frac{1}{2} \left( \frac{m}{2} \right) \left( \frac{m-2}{2} \right) \)

\[ - \sum_{s=3}^{t} \left( \frac{s-1}{2} \right) q_s. \]

PROOF: Equality (i) can be obtained substructing the equalities II and III side by side. Similarly, (ii) can be obtained from II using (i), and (iii) from I using (i) and (ii).

COROLLARY 1. For any hyperbolic plane \( \pi_m \) with \( m \in \{3, 4, 5\} \)

(i) \( q_2 = \frac{1}{2} \left( \frac{m}{2} \right) \left( \frac{m-2}{2} \right) \)

(ii) \( q_1 = \left( \frac{m}{2} \right) \left[ n - 1 - \left( \frac{m-2}{2} \right) \right] \)

(iii) \( q_0 = n^2 + n \left[ 1 - \left( \frac{m}{2} \right) \right] + \left( \frac{m-1}{2} \right) + \frac{1}{2} \left( \frac{m}{2} \right) \left( \frac{m-2}{2} \right). \)

Proof follows from proposition 1 since \( q_i = 0, \ i \geq 3, \) for \( m=4 \) or \( 5 \) and, also \( q_1 = 0 \) for \( m=3. \)

Notice that, if \( m=3 \) then \( q_1 = 3(n-1), q_0 = (n-1)^2. \) Similarly if \( m=4 \) then \( q_1 = 3, q_0 = 6(n-2), q_0 = n^2 - 5n + 6, \) and if \( m=5 \) then \( q_1 = 15, q_0 = 10(n-4), q_0 = n^2 - 9n + 21. \)

The following corollaries are immediate:

COROLLARY 2. Number of lines in \( \pi_0, \pi_1, \pi_2 \) of any hyperbolic plane of type \( \pi_0, \) and \( \pi_2, \) can be determined in terms of the number of lines in \( \pi_3 \) as follows:

\[ q_2 = 45 - 3q_3 \quad q_3 = 105 - 3q_3 \]
\[ q_1 = 15(n-7) + 3q_3 \quad \text{and} \quad q_1 = 21(n-11) + 3q_3 \]
\[ q_0 = n^2 - 14n + 55 - q_3 \quad q_0 = n^2 - 20n + 120 - q_3 \]
respectively.

COROLLARY 3. Total number of lines of \( \pi_0 \) and \( \pi_2 \) in any hyperbolic plane \( \pi_m \) can be determined independently from the number of lines of \( \pi_0 \) and \( \pi_2, \) and vice versa. That is,
\[ q_1 + q_2 = \binom{m}{2} \left[ n - 1 - \frac{1}{2} \binom{m-2}{2} \right] + \frac{1}{2} \left( \sum_{s=3}^{t} s(s-3)q_s \right) \]

\[ q_0 + q_3 = n^2 + n \left[ 1 - \binom{m}{2} + \binom{m-1}{2} \right] + \frac{1}{2} \left( \binom{m}{2} \binom{m-2}{2} \right) - \sum_{s=4}^{t} \binom{s-1}{2} q_s, \]

**THE LINE CLASSES IN \( \pi^0_{n-1} \) AND \( \pi^0_{n-2} \)**

Let \( \pi \) be a projective plane of order \( n \). A set of \( \Theta \) of \( n+1 \) points in \( \pi \) is called an oval if no three points of \( \Theta \) are collinear. A line of \( \pi \) which contains exactly one point, two points and no points of \( \Theta \) is called tangent line, secant line and exterior line, respectively. A point of \( \pi \) is called an exterior point and interior point if it lies on exactly two tangent lines and on no tangent lines, respectively. A secant line contains \( \frac{1}{2} (n-1) \) exterior points and an exterior line contains \( \frac{1}{3} (n+1) \) exterior points. Total number of the exterior points and interior points of \( \pi \) is \( \frac{1}{2} n(n+1) \) and \( \frac{1}{2} n(n-1) \), respectively. There are \( n+1 \) tangent lines of \( \Theta \) and a tangent line contains \( n \) exterior points. Let \( n \) be a projective plane of odd order \( n \), \( n \geq 9 \) and \( \Theta \) an oval in \( \pi \). Let \( \beta \) be the set of interior points of \( \Theta \), and consider the restrictions of the secant and exterior lines of \( \pi \) to the interior points of \( \Theta \). Hence the restrictions of these lines are the set theoretical intersections of the secant and exterior lines of \( \pi \) with \( \beta \). It has been shown by Ostrom [1962] that the geometric structure so obtained is a hyperbolic plane. Clearly the above model of the hyperbolic plane can be considered as a special hyperbolic plane of type \( \pi_m \) provided that \( t_1, t_2, \ldots, t_{n-1} \) are the tangent lines of an oval \( \Theta \). Therefore it will be convenient to use the notation \( \pi^0_{n-1} \) for the Ostrom’s hyperbolic plane. Furthermore, in what follows we use \( \pi^0_m \) instead of \( \pi_m \) provided that the set of deleted lines, \( \{t_1, t_2, \ldots, t_m\} \), with \( 3 \leq m \leq n \), is a subset of the set of all tangent lines of \( \Theta \). It can easily be shown that each of \( \pi^0_3, \pi^0_4, \ldots, \pi^0_{n-3}, \pi^0_{n-1} \) is a hyperbolic plane but not \( \pi^0_n \) since the non-deleted tangent line in \( \pi^0_n \) contains only one point. It is clear from the definitions of corner and exterior points that a corner point for \( \pi^0_m \), \( 3 \leq m \leq n+1 \), is also \( n \) exterior point which is deleted from \( \pi \). It is known that the line classes of \( \pi^0_{n+1} \) are \( C^3_{2(n+1)} \) and \( C^3_{2(n-1)} \).
and \( q_{\frac{1}{2}(n+1)} = \frac{1}{2} \, n(n-1) \) and \( q_{\frac{1}{2}(n-1)} = \frac{1}{2} \, n(n+1) \). We give
line classes of \( \pi^0_{n-1} \) and \( \pi^0_{n-2} \) in the following propositions:

PROPOSITION 2. There exist four line classes in \( \pi^0_{n-1} \), namely \( C_0 \),
\( C_{\frac{1}{2}(n-3)} \), \( C_{\frac{1}{2}(n-3)} \), and the number of lines in these classes are
\[ q_0 = 2, q_{\frac{1}{2}(n-3)} = \frac{1}{2} \, (n-3) \, (n-1), q_{\frac{1}{2}(n-3)} = \frac{1}{2} \, (n-1) \, (n+4), \]
\[ q_{\frac{1}{2}(n-1)} = \frac{1}{2} \, (n+1), \]
respectively.

PROOF. Let \( t_1, t_2 \) be tangent lines of \( \pi^0_{n-1} \) and \( P = t_1 \cap t_2 \), \( Q_i = \theta \cap t_i \).
And let \( Q \) be any point of \( \theta \) with \( Q \neq Q_i \) for \( i = 1, 2 \). Clearly none of the two
lines \( t_1 = PQ_i \) and \( t_2 = PQ_2 \) contains a corner point. Therefore \( t_1 \) and \( t_2 \)
belong to \( C_0 \). Let \( \tau \) be a secant line which passes through none of \( P, Q_1 \)
and \( Q_2 \). All exterior points on \( \tau \) except \( \tau \cap t_1 \) and \( \tau \cap t_2 \) are corner points.

\( \tau \) contains exactly \( \frac{1}{2} \, (n-1)-2 = \frac{1}{2} \, (n-3) \) corner points since
there exist \( \frac{1}{2} \, (n-1) \) exterior points on \( \tau \). Thus \( \tau \) belongs to \( C_{\frac{1}{2}(n-2)} \).

The secant line \( PQ \) contains exactly \( \frac{1}{2} \, (n-1)-1 = \frac{1}{2} \, (n-3) \)
corner points since all exterior points on \( PQ \) except \( P \) are corner points.
Similarly all exterior points on \( Q_1 \, Q \) (or \( Q_2 \, Q \)), except \( Q_1 \, Q \cap t_3 \) (or \( Q_2 \, Q \cap t_3 \)), are corner points. Hence each of the lines \( Q_1 \, Q \) contains
\[ \frac{1}{2} \, (n-1)-1 = \frac{1}{2} \, (n-3) \] corner points. Thus \( PQ, \, Q_1 \, Q \) and \( Q_2 \, Q \)
belong to \( C_{\frac{1}{2}(n-3)} \). Now let \( \tau \) be any line not passing through \( P \). All ex-
terior points on \( \tau \) except \( \tau \cap t_1 \) and \( \tau \cap t_2 \) are corner points. \( \tau \) contains
\[ \frac{1}{2} \, (n+1)-2 = \frac{1}{2} \, (n-3) \] corner points since there exist exactly
\[ \frac{1}{2} \, (n+1) \] exterior points on \( \tau \) in \( \pi \). Thus \( \tau \) belongs to \( C_{\frac{1}{2}(n-3)} \). The se-
cant line \( Q_1 \, Q_2 \) belongs to \( C_{\frac{1}{2}(n-3)} \) since all exterior points on \( Q_1 \, Q_2 \) are
corner points. Finally, an exterior line passing through the point $P$ contains exactly \( \frac{1}{2} \) \((n+1)-1 = \frac{1}{2} \) \((n-1)\) corner points since all exterior points on such a line except $P$ are corner points. Thus, these lines belong to $C_{\frac{1}{2}}(n-1)$. Consequently, the line classes in $\pi^0_{n-1}$ are

\[
C_0 = \{ t_1, t_2 \} \\
C_{\frac{1}{2}}(n-2) = \{ t : t \text{ is a secant line passing through none of } P, Q_i, Q_j \} \\
C_{\frac{1}{2}}(n-3) = \{ t : t = Q_iQ_j, i=1,2, \text{or } t \text{ is a secant line on } P \text{ or an exterior line not on } P \} \\
C_{\frac{1}{2}}(n-1) = \{ t : t = Q_1Q_2 \text{ or } t \text{ is an exterior line on } P \}.
\]

Hence, it is clear that $q_0=2$, $q_{\frac{1}{2}}(n-5) = \frac{1}{2} (n-3) (n-1)$ since the number of secant lines on $P$ is $\frac{1}{2} (n-1)$, the number of secant lines on $Q_1$ or $Q_2$ is $2(n-1)+1$, and the total number of secant lines of $\pi^0_{n-1}$ is $\frac{1}{2} n(n+1)$. $q_{\frac{1}{2}}(n-3) = \frac{1}{2} (n-1)(n+4)$ since the number of secant lines on $P$ is $\frac{1}{2} (n-1)$, the number of secant lines on $Q_1$ or $Q_2$ except $Q_1Q_2$ is $2(n-1)$, and the total number of exterior lines not on $P$ is $\frac{1}{2} (n-1)^2$. $q_{\frac{1}{2}}(n-1) = \frac{1}{2} (n+1)$ since the number of exterior lines on $P$ is $\frac{1}{2} (n-1)$, and $Q_1Q_2 \in C_{\frac{1}{2}}(n-1)$.

**Proposition 3.** There exist four line classes in $\pi^0_{n-2}$, namely $C_0$, $C_{\frac{1}{2}}(n-\gamma)$, $C_{\frac{1}{2}}(n-\delta)$, $C_{\frac{1}{2}}(n-\mu)$, and the number of lines in these classes are

\[
q_0=3, \quad q_{\frac{1}{2}}(n-\gamma) = \frac{1}{2} (n-3)(n-5), \quad q_{\frac{1}{2}}(n-\delta) = \frac{1}{2} (n+8)(n-3), \\
q_{\frac{1}{2}}(n-\mu) = \frac{3}{2} (n+3), \text{ respectively.}
\]
SKETCH OF PROOF. Let $t_1, t_2, t_3$ be non deleted tangent lines and $t_1 \cap t_2 = P_3$, $t_1 \cap t_2 = P_2$, $t_2 \cap t_3 = P_1$; and let $t_i \cap \emptyset = Q_i$ with $i = 1, 2, 3$. One can find the line classes of $\pi^{0}_{n-2}$ as follows:

$C_0 = \{t_1, t_2, t_3\}$

$C_{\frac{1}{2}(n-1)} = \{t: t \text{ is a secant line passing through none of } P_1, Q_1 \text{ with } i = 1, 2, 3\}$

$C_{\frac{1}{2}(n-3)} = \{t: t \text{ is a secant line on } P_1 \text{ but not on } Q_i \text{ or a secant line passing through only one } Q_i \text{ but none of } P_1 \text{ or } t \text{ is an exterior line not on } P_1, i = 1, 2, 3\}$

$C_{\frac{2}{3}(n-3)} = \{t: t = P_1 Q_i \text{ or } t = Q_i Q_j \text{ with } i \neq j \text{ or } t \text{ is an exterior line on } P_1, i = 1, 2, 3\}$.

Proof can be completed by a similar way in the proof of proposition 2.

REFERENCES


