ON 6– FIGURES IN MOUFANG PROJECTIVE PLANES

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ABSTRACT

In this paper, the concept of cross-ratio is extended to the whole Moufang plane and some properties of 6–figures in the π Moufang plane is examined. Essentially the geometric properties of the π which is equivalent to the existence of a square root of an element of an alternative division ring R is determined.

INTRODUCTION

Let Π be a Moufang projective plane. It is well known that π determines a unique alternative ring R. One of the main problems in projective geometry is to find geometric properties of π which are equivalent to certain algebraic properties of the alternative ring R. For instance, Π is Desarguesian if and only if R is associative. In this paper, we extend some of the properties of the 6–figures, which have been given by Cater [1] for Desarguesian planes, to the Moufang planes. Namely, we determined the geometric properties of Π equivalent to the existence of a square root of an element of R. And if (A,B:C,D) denotes the cross-ratio of distinct collinear points A,B,C,D in Π, we construct points J and N such that (A,B:C,J) = (A,B:C,D)² and (A,B:C,N) = (A,B:C,D)³. In fact, these are investigated in [2]. But some mappings which is obtained by using the identity

\[ x^{-1}(y(xz)) = (x^{-1}yx)z \]

asserted in [5] for Cayley-Dickson algebras are used. However [3] demonstrated here that (1) does not valid for Cayley-Dickson algebras. But also the existence of 4–point transitivity is shown by using of some new mappings. Essentially, in this paper, [2] is rearranged and modified under the light of [3].
Throughout the paper we use the terminology in [1] and [3]; A 6–figure is a sequence of 6 distinct points (ABC, A’B’C’) such that ABC is a triangle, and A’eBC, B’eCA, C’eAB. The points A,B,C,A’,B’,C’ are called vertices of this 6–figure. A 6–figure (ABC, A’B’C’) is said to be equivalent to any 6–figure (DEF,D’E’F’) if there exist a projective collineation of II which maps A,B,C,A’,B’,C’ to D,E,F,D’,E’,F’ respectively; in symbols ABCA’B’C’ ~ DEF,D’E’F’.

(ABC, A’B’C’) is called a menelaus 6–figure if A’,B’,C’ are collinear; and (ABC,A’B’C’) is called a ceva 6–figure if the lines AA’,BB’,CC’ are concurrent.

Throughout the paper, we assume that R is an alternative division ring with center K of arbitrary characteristic and that II is the Moufang plane coordinatized by R, [4].

It is easy to see that the mappings
\[ I_1:(x,y) \rightarrow (x^{-1}, yx^{-1}), \quad x \neq 0, (0,0) \leftrightarrow (y), \quad (\infty) \rightarrow (\infty) \]
\[ [m,k] \rightarrow [k,m], \quad [k] \rightarrow [k^{-1}], \quad k \neq 0, [0] \leftrightarrow [\infty] \]
and
\[ I_2:(x,y) \rightarrow (xy^{-1}, y^{-1}), \quad y \neq 0, (x,0) \leftrightarrow (x^{-1}), \quad x \neq 0, (0,0) \leftrightarrow (\infty), \quad (0) \rightarrow (0) \]
\[ [m,k] \rightarrow [-k^{-1}m,k^{-1}], \quad k \neq 0, [m,0] \leftrightarrow [m^{-1}], m \neq 0, [0,0] \leftrightarrow [\infty], [0] \rightarrow [0] \]
are collineations of II.

Let \( A = (0), \quad B = (\infty), \quad C = (0,0), \quad A' = (0,1), \quad B' = (-1,0), \quad C' = (-m) \)
for some \( m \in \mathbb{R} \). Here, \((-1,0) \leftrightarrow (\infty), \quad (0,1) \leftrightarrow (0) \) and
\[ I_2I_1:(0),(\infty),(0,0),(-1,1) \rightarrow (\infty), (0,0), (0), (1,-1). \]
According to the Theorem 1 in [3], the mapping \( g \) which maps \((0),(\infty), (0,0),(1,-1), \) to \((0),(\infty),(0,0),(-1,m)\) is a composition of the collineations \( F_e \) and \( S_{\alpha,\beta} \) where \( e \in \mathbb{R}, \quad \alpha,\beta \in \mathbb{K} \). We already known that such a collineation, maps \((x,y)\) to \((x',yd)\). Now, since \( g \) maps \((1,-1)\) to \((-1,m)\) than \((-1) \rightarrow m \) and so \(-m^{-1} = d^{-1} \).

Therefore \((0,-m^{-1})\) maps to \((0,-m^{-1} d) = (0,1)\). Thus
\[ f = gl_1 \circ l_2 \circ (0),(\infty),(0,0),(0,1),(-1,0),(-m) \rightarrow (\infty),(0,0),(0),(1,0),(-m),(0,1). \]
Consequently
\[ ABCA'B'C' \sim BCAB'C'A' \sim CAB'C'A'B' \]
is shown by using \( f \) and \( f^2 \).
Furthermore, any 6-figure \((DEF, D'E'F')\) is equivalent to 
\(((0)(\infty)(0,0), (0,1)(-1,0)(-m))\) for some \(m \in \mathbb{R}\); since there exist by Theorem 1 in [3], a collineation mapping \(D,E,D',E'\) 4-point on \((0), (\infty), (0,1), (-1,0)\). Thus \((ABC, A'B'C'), (BCA, B'C'A'), (CAB, C'A'B')\) will be regarded as the same 6-figure \(\mu\) and likewise \((ACB, A'C'B'), (CBA, C'B'A'), (BAC, B'A'C')\) as the same 6-figure \(\lambda\). \(\mu\) and \(\lambda\) are called opposite 6-figures of each other; in symbols, \(\lambda = \mu^{-1}\) and \(\mu = \lambda^{-1}\).

Let \(\Pi\) be a Moufang plane satisfying the Fano's Axiom. It follows, (see [7]) that there exist unique points \(A' \in BC, B' \in CA, C' \in AB\) such that \(H(AB, C'C'), H(BC, A'A''), H(CA, B'B'\)). The 6-figure \((ABC, A'B'C')\) is called the conjugate of \(\mu\), in symbol \(-\mu\). Likewise \(\mu\) is the conjugate of \(-\mu\).

Let \(C \in AB\) be the point such that \(C, C'\) and \(AA', BB'\) are collinear. Let \(A' \in BC\) and \(B' \in CA\) be the points such that \(A, A'\) and \(BB', CC'\) are collinear and \(B, B'\) and \(AA', CC'\) are collinear. The 6-figure \((ACB, A'B'C')\) is called the first descendant of \(\mu\), written \(\mu^d\). \(\mu\) is called a first ancestor of \(\mu^d\).

Let \(A' = BC, B' = CA, C' = AB, A'B'\). The figure \((ACB, A'C'B')\) is called the first codecent of \(\mu\), written \(\mu^c\). \(\mu\) is called a first coancestor of \(\mu^c\).

Figure 1 represents the 6-figure \(\mu_1 = ((0)(\infty)(0,0), (0,1)(-1,0)(-m))\) and \(\mu_1^c\) and construction of \(B^{cc}\). In figure 2 \(\mu_1, \mu_1^d\) are drawn and \(B^{dc}\) is constructed. In figure 3 the points \(A', B', C', B^c\) are drawn.
SOME PROPERTIES OF 6-FIGURES IN MOUFANG PLANES

The classical definition of the cross-ratio for Desarguesian planes is not available for the Moufang planes. Ferrar [5] gives the following algebraic definition of the cross-ratio for the points on the line \([0,0]\).

\[(A,B:C,D) = (a,b:c,d) = \{(a-d)^{-1} (b-d) \ (b-e)^{-1} (a-e)\}\]

where \(A = (a,0)\), \(B = (b,0)\), \(C = (c,0)\), \(D = (d,0)\) and \(\{x\}\) denotes the conjugancy class of \(x\) in the alternative ring \(R\). We extend this definition to whole plane as follows:
(i) If \( A = (a,a), B = (b,b), C = (c,c), D = (d,d) \) are on a line of type \([m,k]\) let \((A,B;C,D) = (a,b;c,d)\). For this case one of \(A,B,C,D\) is the ideal point \((m)\), use \(\infty \notin \mathbb{R}\) instead of the corresponding component in \((a,b;c,d)\). (Notice the perspectivity from \([m,k]\) to \([0,0]\) with center \((\infty)\)).

(ii) If \(A,B,C,D\) are on a line of type \([k]\) that is, \(A = (k,a), B = (k,b), C = (k,c), D = (k,d)\) let \((A,B;C,D) = (a,b;c,d)\). In the case one of \(A,B,C,D\) is \((\infty)\) use again \(\infty\) in \((a,b;c,d)\). (Notice that the perspectivity from \([k]\) to \([0,0]\) with center \((0,1)\) maps \((k,x) \to (k+x,0)\); and taking \(k+x\) instead of \(x\) does not change the cross ratio.\).

(iii) If the points are on \([\infty]\), that is \(A = (a), B = (b), C = (c), D = (d)\), let \((A,B;C,D) = (a,b;c,d)\). (Notice that the perspectivity from \([\infty]\) to \([0,0]\) with center \((0,-1)\) maps \((m) \to (m^{-1},0), (0) \to (0), (\infty) \to (0,0), \) and \((a,b;c,d) = (a^{-1}, b^{-1}, c^{-1}, d^{-1})\).)

**Lemma 1.** In a Moufang plane a perspectivity preserve the cross ratio.

**Proof:** Let \(A,B,C,D \in \mathbb{P}, \varnothing_\mathbb{M}\) and \(\varnothing_\mathbb{N}\) be perpectivities from any line \([0,0]\) which have center \(M\) and \(N\), respectively. Where \(\varnothing_\mathbb{M}\) is the perspectivity in defination of generalized cross - ratio, and \(N \neq M, N \notin \mathbb{P}, N \in [0,0]\). In addition let \(\varnothing_\mathbb{M}: A,B,C,D \to A_1,B_1,C_1,D_1\) and \(\varnothing_\mathbb{N}: A,B,C,D \to A_0,B_0,C_0,D_0\).

To prove, it is sufficient to show that \(\varnothing_\mathbb{N}\) preserves the cross-ratio. According to the definition \((A,B;C,D) = (A_1,B_1;C_1,D_1)\). Thus
\[
\varnothing = \varnothing_\mathbb{M} \varnothing_\mathbb{N}^{-1}: A_0,B_0,C_0,D_0 \to A_1,B_1,C_1,D_1
\]

Therefore \(\varnothing\) is a projectivity of \([0,0]\).

Since \(\varnothing\) is a projectivity of \([0,0]\) which preserves the cross-ratio (Sec [5], Theorem 3,7) \((A_1,B_1;C_1,D_1)\) is equal to \((A_0,B_0;C_0,D_0)\). Consequently \((A,B;C,D) = (A_0,B_0;C_0,D_0)\), that is, \(\varnothing_\mathbb{N}\) preserves the cross-ratio.

In what follows we use the fact that distinct collinear points \(A,B,C,\) \(D\) in \(\Pi\) are in harmonic position if and only if \((A,B;C,D) = -1\).

**Lemma 2.** If \(\mu_1 = (ABC,A'B'C') = ((0) (\infty) (0,0), (0,1) (-1,0) (-m))\) then
\[(A,B;C',C) = (B,C;A',A) = (C,A;B',B) = \{m\}.
\]

**Proof:** is trivial.
Conjugacy class of \(-\langle A,B;C',C \rangle = \langle C,A;B',B \rangle\) is called the ratio of the 6-figure \(\mu = \langle ABC,A'B'C' \rangle\), denoted by \(r(\mu)\).

**THEOREM 1:**

(i) \(\mu\) is a menelaus 6-figure if and only if \(r(\mu) = -1\)

(ii) \(\mu\) is a ceva 6-figure if and only if \(r(\mu) = 1\)

Proof: (i) It suffices to assume that \(\mu\) is \(\mu_1\) because cross-ratio is preserved by projective collineations. Thus \(\mu_1 = ((0)(\infty)(0,0),(0,1)(-1,0)(1))\) and \(r(\mu_1) = -1\).

Conversely, if \(r(\mu_1) = -1\); then \(\mu_1 = ((0)(\infty)(0,0),(0,1)(-1,0)(1))\) and the points of \((0,1),(-1,0),(1)\) are collinear.

(ii) If \(\mu_1\) is a ceva 6-figure, then \(\mu_1 = ((0)(\infty)(0,0),(0,1)(-1,0)(1))\) and \(r(\mu_1) = 1\).

Conversely, if \(r(\mu_1) = 1\), \(\mu_1\) as above thus, the lines of \((0)(0,1),(
\infty)(-1,0),(0,0)(-1)\) is concurrent.

**THEOREM 2:** For any 6-figure \(\mu\) we have

(i) \(r(\mu^{-1}) = (r(\mu))^{-1}\)

(ii) \(r(-\mu) = -r(\mu)\)

(iii) \(r(\mu^d) = (r(\mu))^2\)

(iv) \(r(\mu^c) = -(r(\mu))^2\)

Proof: It is sufficient to assume that \(\mu\) is \(\mu_1\) in figures 1,2,3 since cross ratio is preserved by projective collineations. With a simple calculation we have

(i) \(r(\mu_1^{-1}) = -(\infty,0:-1,m^{-1}) = \{m^{-1}\} = (r(\mu_1))^{-1}\);

(ii) \(r(-\mu_1) = -(0,\infty:1,m^{-1}) = \{-m\} = -(r(\mu_1))\);

(iii) \(r(\mu_1^d) = -(\infty,0:-m^{-1},m) = \{m^2\} = (r(\mu_1))^2\)

(iv) \(r(\mu_1^c) = -(\infty,0:m^{-1},m) = \{-m^2\} = -(r(\mu_1))^2\);
It follows immediately, from theorem 1 and 2 that conjugate of a menelaus 6-figure is ceva 6-figure and vice versa. In fact this has been shown in [6].

THEOREM 3. Let $m \in \mathbb{R}$, $m \neq 0$. Then the equation $x^2 = m$ (or $x^2 = -m$) has a solution in $\mathbb{R}$ if and only if any 6-figure $\mu$ with ratio \{m\} has a first ancestor (coancestor) in $\Pi$.

Proof: Without loss of generality we take $\mu = \mu_1$. Let $\lambda = ((0) (0,0) (\infty), DEF)$ be a first ancestor of $\mu_1$. Then we have \{q^2\} = \{m\} by Theorem 1, and there exists an $x \in \mathbb{R}$ satisfying $x^2 = m$.

Conversely suppose that there exist $q \in \mathbb{R}$ such that $q^2 = m$. In this case it is easily computed that $\lambda_1 = ((0) (0,0) (\infty), (0,-q)(q)(q^{-1},0))$ is a first ancestor of $\mu_1$.

Now consider the equation $x^2 = -m$. Let $\lambda_1 = ((0)(0,0)(\infty), STU)$ be a first coancestor of $\mu_1$ and let $r(\lambda_1) = \{q\}$. Then we have \{m\} = \{-q^2\} by Theorem 1, and there exists an $x \in \mathbb{R}$ satisfying $x^2 = -m$.

Suppose $q \in \mathbb{R}$ and $q^2 = -m$. Then one can easily show that $\lambda_1 = ((0)(0,0)(\infty), (0,q)(q)(-q^{-1},0))$ is a first coancestor of $\mu_1$.

Clearly one can easily observe from the figures that first descendant of $\mu$ and $-\mu$ are same, and that first codescendant of $\mu$ and $-\mu$ are same. Namely $\mu^d = (-\mu)^d$ and $\mu^c = (-\mu)^c$.

Finally, it is worth to note that the construction which gives an algorithm for "squaring" a cross-ratio of points in a Desarguesian plane ([1]) can be also extended to a Moufang plane as follows:

Let $A,B,C,D$ be any collinear points in $\Pi$. Choose any point $E$ not on $AB$, and choose points $F \in AE, G \in BE$ such that $F,G$ and $C$ are collinear points.

Let $I = AE \cdot DG$, $H = BE \cdot DF$ and $J = AB \cdot HI$. For this project $A,B,F,G$ to $(0),(0,0),(-1),(0,1)$ respectively. Thus $(A,B;C,D) = \{m\}$ and $(A,B;C,J) = \{m^2\}$ by Lemma 1, and consequently $(A,B;C,J) = (A,B;C,D)^2$.

Furthermore, if $K = AE \cdot CH, M = AB \cdot KG$ and $G' = BE \cdot MF, I' = AE \cdot G'D, N = AB \cdot HI'$ then $(A,B;C,N) = (A,B;C,D)^3$. 
REFERENCES


