ON SOME GENERALIZED SEQUENCE SPACES

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ABSTRACT

In this paper we introduce a general sequence space \( \Delta_\gamma(X) = \{x = (x_k): (v_kx_k - v_{k+1}x_{k+1}) \in X\} \), where \( X \) is any sequence space. We establish some inclusion relations, topological results, in general case and we characterize the continuous, \( \alpha-, \beta- \) and \( \gamma- \) duals of \( \Delta_\gamma(X) \) for various sequence spaces \( X \). The results of this paper, in a particular case, include the corresponding results of KIZMAZ.

1. INTRODUCTION

Let \( v = (v_k) \) be any fixed sequence of nonzero complex numbers satisfying

\[
\lim_{k} \inf |v_k|^{1/k} = r \quad (0 < r \leq \infty).
\]  

(1.1)

Define a function \( \Lambda: C \to C \), where \( C \) denotes the set of complex numbers, by

\[
\Lambda(z) = \frac{z}{\sum_{k=1}^{k=\infty} \frac{z^k}{v_k}}.
\]  

(1.2)

(Throughout this paper, \( \sum \) will mean summation from \( k = 1 \) to \( k = \infty \).)

Obviously \( \Lambda \) is an analytic function in the disc \( E_r = \{z: |z| \leq r\} \) because of (1.1).

Now let \( X \) be any sequence space of complex numbers. Then we define

\[
\Delta^\alpha(X) = \{f: f(z) = \sum_{k} x_kz^k \text{ such that } \Delta_\nu(x) \in X\},
\]  

(1.3)

where \( \Delta_\nu(x) = (\Delta_\nu(x_k)) = (v_kx_k - v_{k+1}x_{k+1}) \).

\[\text{ISSN 02571-081 A.Ü. Basimevi}\]
We can always get a sequence space
\[ \Delta_v(X) = \{x = (x_k) : \Delta_v(x) \in X \}. \] (1.4)

It is easy to show that there exists an algebraic isomorphism between \( \Delta^A(X) \) and \( \Delta_v(X) \) in the sense that \( f \to x = (x_k) \) is an algebraic isomorphism. Therefore \( \Delta_v(X) \) also can be regarded as a set of functions.

Let \( \ell_\infty \), \( c \) and \( c_0 \) be the linear spaces of bounded, convergent and null sequences \( x = (x_k) \), respectively, normed by
\[ \|x\|_\infty = \sup_k |x_k| \quad (k = 1,2,\ldots). \]

Now we define \( \Delta^A(\ell_\infty) \), \( \Delta^A(c) \) and \( \Delta^A(c_0) \) as follows:
\[ \Delta^A(\ell_\infty) = \{f : f(z) = \sum_k x_k z^k \text{ such that } \sup_k |\Delta_v(x_k)| < \infty \} \]
\[ \Delta^A(c) = \{f : f(z) = \sum_k x_k z^k \text{ such that } \Delta_v(x_k) \to \iota \text{, for some } \iota, \text{ as } k \to \infty \} \]
\[ \Delta^A(c_0) = \{f : f(z) = \sum_k x_k z^k \text{ such that } \Delta_v(x_k) \to 0 \text{ as } k \to \infty \}. \]

All these classes contain those analytic functions which are analytic in the disc \( E_R = \{z : |z| \leq R, \text{ where } R \gg r\} \).

If \( f(z) = \sum_k x_k z^k \), belongs to \( \Delta^A(\ell_\infty) \) then the coefficients \( x_k \) (\( k = 1,2,\ldots \)) satisfy the following conditions:
\[ (i) \sup_k k^{-1} |v_k x_k| < \infty, \]
\[ (ii) \sup_k |v_k x_k - k(k+1)^{-1} v_{k+1} x_{k+1}| < \infty. \]

And conversely if (i) and (ii) hold, then \( f \in \Delta^A(\ell_\infty) \) (see Lemma 2).

Suppose that \( f(z) = \sum_k x_k z^k \) and \( f \in \Delta^A(\ell_\infty) \) so that, from (i), we have
\[ |x_k|^{1/k} \leq \frac{K^{1/k} k^{1/k}}{|v_k|^{1/k}} \]
for some \( K > 0 \) and for every \( k = 1,2,\ldots \). Hence, by (1.1), we obtain
\[ \frac{1}{R} = \lim_k \sup_k |x_k|^{1/k} \leq \frac{1}{\lim_k \inf |v_k|^{1/k}} = \frac{1}{r}. \]
which implies that $R \gg r$. Again if $f \in \Delta^A (\iota_{\infty})$, then condition (ii) holds which implies that

$$ k^{-2} |v_k x_k| \leq K_1 $$

for some $K_1 > 0$ and for every $k = 1, 2, \ldots$. A similar line of reasoning will lead to $R \gg r$. So the class $\Delta^A (\iota_{\infty})$ contains those analytic functions which are analytic in the disc $E_R$.

In a similar way, it can be shown that the classes $\Delta^A (e)$ and $\Delta^A (e_0)$ contain those analytic functions which are analytic in the disc $E_R$.

Taking into account the algebraic isomorphism, we view $\Delta^A (\iota_{\infty})$, $\Delta^A(e)$ and $\Delta^A (e_0)$ as sequence spaces $\Delta_v (\iota_{\infty})$, $\Delta_v (e)$ and $\Delta_v (e_0)$, respectively, which are defined as follows:

$$ \Delta_v (\iota_{\infty}) = \{ x = (x_k) : \sup_k |\Delta_v (x_k)| < \infty \}, $$

$$ \Delta_v (e) = \{ x = (x_k) : \Delta_v (x_k) \to \iota, \text{ for some } \iota \text{, as } k \to \infty \}, $$

$$ \Delta_v (e_0) = \{ x = (x_k) : \Delta_v (x_k) \to 0 \text{ as } k \to \infty \}. $$

If we consider $(v_k) = (1,1,\ldots)$ in (1.4), then $\Delta_v (X)$ becomes $\Delta (X)$, where

$$ \Delta (X) = \{ x = (x_k) : (x_k - x_{k+1}) \in X \} $$

which was studied by KizmaZ [2] for $X = \iota_{\infty}$, $e$ and $e_0$. The results of this paper, in a particular case, include the corresponding results of his.

2. SOME PROPERTIES OF $\Delta_v (X)$

In this section, we give some relations between $\Delta_v (X)$ and $X$, and we discuss some topological properties of $\Delta_v (X)$.

**Theorem 1:** If $X$ is a Banach space normed by $\| \|$ , then $\Delta_v (X)$ is also a Banach space normed by

$$ \| x \|_{\Delta} = |v_1 x_1| + \| \Delta_v (x) \|. \quad (2.1) $$

**Proof:** Since $(0) \in \Delta_v (X)$, $\Delta_v (X) \neq \emptyset$. Clearly, $\Delta_v (X)$ is a linear space. It is easy to show that $\Delta_v (X)$ is a normed space with norm defined in (2.1).
Now we show that $\Delta_v(X)$ is complete. Let $(x^n)$ be a Cauchy sequence in $\Delta_v(X)$, where $x^n = (x_1^n, x_2^n, \ldots) \in \Delta_v(X)$. Then

$$\|x^m - x^n\|_\Delta \to 0 \text{ as } m, n \to \infty,$$

that is,

$$\|(x_k^m - x_k^n)\|_\Delta \to 0 \text{ as } m, n \to \infty.$$

Hence,

$$|x_i^m - x_i^n| + \|\Delta_v(x^m) - \Delta_v(x^n)\| \to 0 \text{ as } m, n \to \infty.$$

Therefore $(x_1^1, x_1^2, \ldots)$ and $(\Delta_v(x^1), \Delta_v(x^2), \ldots)$ are Cauchy sequences in C and X respectively. Since C and X are complete, they are convergent. Suppose that $x_i^n \to x_i$ in C and $(\Delta_v(x^n)) \to (z_k)$ in X, as $n \to \infty$.

Let $z_k = \Delta_v(x_k)$ so that $x_k = \sum_{i=1}^k z_{i-1}$. Then $(\Delta_v(x^n))$

$= ((\Delta_v(x_1^1)), (\Delta_v(x_2^1)), \ldots)$ converges to $(\Delta_v(x_k))$ in X. Hence,

$$\|x^n - x\|_\Delta \to 0 \text{ as } n \to \infty.$$

Therefore, $\Delta_v(X)$ is complete. Consequently it is a Banach space.

**Lemma 1:** If $X \subset Y$, then $\Delta_v(X) \subset \Delta_v(Y)$.

**Proof:** It is trivial.

**Theorem 2:** Let X be a Banach space and A, a closed subset of X. Then $\Delta_v(A)$ is also closed in $\Delta_v(X)$.

**Proof:** Since $A \subset X$, $\Delta_v(A) \subset \Delta_v(X)$ by Lemma 1. Now let $x \in \overline{\Delta_v(A)}$, the closure of $\Delta_v(A)$. Then there exists a sequence $(x^n)$ in $\Delta_v(A)$ such that

$$\|x^n - x\|_\Delta \to 0 \text{ as } n \to \infty$$

in $\Delta_v(A)$. Hence,

$$\|(x_k^n) - (x_k)\|_\Delta \to 0 \text{ as } n \to \infty$$

in $\Delta_v(A)$ so that

$$|x_1^n - x_1| + \|\Delta_v(x_k^n) - \Delta_v(x_k)\| \to 0 \text{ as } n \to \infty$$

in A. Thus $\Delta_v(x) \in \overline{A}$ which implies that $x \in \Delta_v(\overline{A})$. Conversely, if $x \in \Delta_v(\overline{A})$, then $x \in \Delta_v(A)$. Therefore, $\Delta_v(A) = \Delta_v(\overline{A})$ and since A is closed, $\overline{\Delta_v(A)} = \Delta_v(A)$. Consequently, $\Delta_v(A)$ is a closed subset of $\Delta_v(X)$.
**Theorem 3:** If $X$ is a separable space, then $\Delta_v(X)$ is also a separable space.

**Proof:** Let $X$ be a separable space. Then there exists a countable subset $A$ of $X$ such that $\bar{A} = X$. Since $\bar{A} = X$, then $\Delta_v(A) = \Delta_v(\bar{A}) = \Delta_v(X)$ which can be easily shown by similar arguments as in the proof of Theorem 2. If we define $f : \Delta_v(A) \to A$ by $f(x) = \Delta_v(x)$ for $x \in \Delta_v(A)$, then it is clear that $f$ is bijective. Therefore $\Delta_v(A)$ is countable, since $A$ is countable. Hence $\Delta_v(A)$ is a countable subset of $\Delta_v(X)$ such that $\bar{\Delta_v(A)} = \Delta_v(X)$. Consequently, $\Delta_v(X)$ is separable.

**Theorem 4:** In general, $\Delta_v(X)$ need not be a sequence algebra.

**Proof:** To prove this, we give a counter example. It is well-known that $c_0$ is sequence algebra. Let $x = y = (\sqrt{k})$. Clearly, $x,y \in \Delta_v(c_0)$, if we choose $(v_k) = (1,1,\ldots)$; but $z \notin \Delta_v(c_0)$, since $\Delta_v(z) = (x_ky_k - x_{k+1}y_{k+1}) = (-1,-1,\ldots)$, where $z = (x_ky_k)$. This completes the proof.

**Corollary 1:** (i) $\Delta_v(t_\infty)$ is a BK-space with norm defined by

$$\|x\|_\Delta = \|v_1x_1\| + \sup_k |\Delta_v(x_k)|.$$

(ii) $\Delta_v(c)$ and $\Delta_v(c_0)$ are separable BK-spaces with the norm as in (2.2).

**Proof:** Since $t_\infty$, $c$ and $c_0$ are Banach spaces, then $\Delta_v(t_\infty)$, $\Delta_v(c)$ and $\Delta_v(c_0)$ are also Banach spaces by Theorem 1. Since $c$ and $c_0$ are separable spaces, then $\Delta_v(c)$ and $\Delta_v(c_0)$ are separable spaces by Theorem 3.

Since $\|x^n - x\|_\Delta \to 0$, as $n \to \infty$ in $\Delta_v(t_\infty)$, implies that $|x_k^n - x_k| \to 0$, for each $k = 1,2,\ldots$, as $n \to \infty$, it follows that $\Delta_v(t_\infty)$ is also a BK-space, since it is a Banach space with continuous coordinates.

It is easy to show that $\Delta_v(c)$ and $\Delta_v(c_0)$ are BK-spaces.

Assuming $(v_k) = (1,1,\ldots)$ in Corollary 1, we obtain the following results.

**Corollary 2:** (i) $\Delta(t_\infty)$ is a BK-space with norm

$$\|x\| = \|x_1\| + \sup_k |x_k - x_{k+1}|.$$  

(ii) $\Delta(c)$ and $\Delta(c_0)$ are separable BK-spaces with the same norm as in (2.3).
Remark: It may be found in [2] that $\Delta(\ell_\infty)$, $\Delta(c)$ and $\Delta(c_0)$ are BK-spaces.

Now let us define

$$D : \Delta_v(X) \to \Delta_v(X)$$

by $D(x) = y = (0, x_2, x_3, \ldots)$, where $X$ stands for $\ell_\infty$, $c$ or $c_0$. Then $D$ is a bounded linear operator on $\Delta_v(X)$ and $\|D\| = 1$. Further,

$$D[\Delta_v(X)] = \Delta'_v(X) = \{x = (x_k) : x \in \Delta_v(X), x_1 = 0\} \subseteq \Delta_v(X)$$

is a subspace of $\Delta_v(X)$ and

$$\|x\|_e = \|\Delta_v(x)\|_e$$

in $\Delta'_v(X)$. $\Delta'_v(X)$ and $X$ are equivalent as topological spaces [3], since

$$T : \Delta'_v(X) \to X,$$ defined by, $Tx = y = (\Delta_v(x_k))$ \hspace{1cm} (2.4)

is a linear homeomorphism. Also $T$ and $T^{-1}$ are norm preserving and $\|T\| = \|T^{-1}\| = 1$.

Now let $(\Delta'_v(x))^*$ and $X^*$ denote the continuous duals of $\Delta'_v(X)$ and $X$ respectively. Then

$$S : (\Delta'_v(X))^* \to X^*,$$

defined by

$$f_T \to f = f_T \circ T^{-1},$$

is a linear isometry, where $X$ stands for $\ell_\infty$, $c$ or $c_0$. Thus, $(\Delta'_v(X))^*$ is equivalent to $X^*$. Therefore,

$$(\Delta'_v(\ell_\infty))^* \simeq \ell_\infty^*$$

and

$$(\Delta'_v(c))^* \simeq (\Delta'_v(c_0))^* \simeq \ell_1,$$

since $c^* \simeq c_0^* \simeq \ell_1$, where $\ell_1 = \{x = (x_k) : \Sigma_k |x_k| < \infty\}$ [1].

3. KÖTHE–TOEPLITZ DUALS OF $\Delta_v(X)$

In this section, we characterize the $\alpha$-, $\beta$- and $\gamma$- duals of $\Delta_v(\ell_\infty)$ and $\Delta_v(c)$. 

Definition ([1]): Let \( X \) be a sequence space and define

(i) \( X^\alpha = \{ a = (a_k) : \sum_k |a_k x_k| < \infty \text{ for all } x \in X \} \),

(ii) \( X^\beta = \{ a = (a_k) : \sum_k a_k x_k \text{ converges for all } x \in X \} \),

(iii) \( X^\gamma = \{ a = (a_k) : \sup_n \left| \sum_{k=1}^n a_k x_k \right| < \infty \text{ for all } x \in X \} \).

Then \( X^\alpha, X^\beta \) and \( X^\gamma \) are, respectively, called the \( \alpha \)-, \( \beta \)- and \( \gamma \)-dual spaces of \( X \). \( X^\alpha \) is also called Köthe-Toeplitz dual space and \( X^\beta \) is also called generalised Köthe-Toeplitz dual space. It is easy to show that \( \emptyset \subset X^\alpha \subset X^\beta \subset X^\gamma \). If \( X \subset Y \), then \( Y^\gamma \subset X^\gamma \) for \( \gamma = \alpha, \beta \) or \( \gamma \).

Lemma 2. The following conditions (1) and (2) are equivalent.

1. \( \sup_k |\Delta_v(x_k)| < \infty \),

2. (i) \( \sup_k k^{-1} |v_k x_k| < \infty \)

(ii) \( \sup_k |v_k x_k - k(k+1)^{-1} v_{k+1} x_{k+1}| < \infty \).

Proof: Suppose that Condition 1 holds. Then

\[
|v_1 x_1 - v_{k+1} x_{k+1}| = |\sum_{i=1}^k \Delta_v(x_i)| \leq \sum_{i=1}^k |\Delta_v(x_i)| = o(k)
\]

which implies that 2(i) holds. Since

\[
|v_k x_k - k(k+1)^{-1} v_{k+1} x_{k+1}| = |k(k+1)^{-1} \Delta_v(x_k)| +
\]

\[
(k+1)^{-1} |v_k x_k| \leq |\Delta_v(x_k)| + (k+1)^{-1} |v_k x_k|,
\]

we obtain 2(ii).

Now suppose that Condition 2 holds. Then from the inequality

\[
|v_k x_k - k(k+1)^{-1} v_{k+1} x_{k+1}| \geq k(k+1)^{-1} |\Delta_v(x_k)| -
\]

\[
(k+1)^{-1} |v_k x_k|,
\]

we obtain Condition 1.

In the following lemmas, \( (p_n) \) will denote a sequence of positive numbers increasing monotonically to infinity.
Lemma 3 ([2]): (i) If \( \sup_n | \sum_{k=1}^n p_k a_k | < \infty \), then
\[
\sup_n | p_n \sum_{k=n+1}^\infty a_k | < \infty.
\]

(ii) If \( \sum_k p_k a_k \) is convergent, then \( \lim_n p_n \sum_{k=n+1}^\infty a_k = 0 \).

Lemma 4: (i) If \( \sup_n | \sum_{k=1}^n p_k v_k^{-1} a_k | < \infty \), then
\[
\sup_n | p_n \sum_{k=n+1}^\infty v_k^{-1} a_k | < \infty.
\]

(ii) \( \sum_k k v_k^{-1} a_k \) is convergent, if and only if \( \sum_n b_n \) is convergent with \( \lim_n n b_n = 0 \), where \( b_n = \sum_{k=n+1}^\infty v_k^{-1} a_k \).

Proof: (i) If we put \( v_k^{-1} a_k \) instead of \( a_k \) in Lemma 3(i), the result is immediate.

(ii) If we put \( p_n = n \) and choose \( v_k^{-1} a_k \) instead of \( a_k \), the result follows from Lemma 3(ii), since
\[
\sum_{k=1}^n k v_k^{-1} a_k = \sum_{k=1}^n k (b_{k-1} - b_k) = \sum_{k=1}^n b_{k-1} - n b_n.
\]

Now we give the main theorem which characterizes the \( \alpha \)-, \( \beta \)- and \( \gamma \)- duals of \( \Delta_v(X) \), where \( X \) stands for \( \iota_\infty \) or \( c \).

Theorem 5: Let \( X \) stand for \( \iota_\infty \) or \( c \). Then,

(i) \( (\Delta_v(X))^\alpha = \{ a = (a_k) : \sum_k k |v_k^{-1} a_k| < \infty \} \),

(ii) \( (\Delta_v(X))^\beta = \{ a = (a_k) : \sum_k k v_k^{-1} a_k \text{ converges and } \sum_k |b_k| < \infty \} \),
(iii) \((\Delta_v(X))^\gamma = \{a = (a_k): \sup_n \sum_{k=1}^n kv_k^{-1}a_k < \infty \text{ and} \sum_k |b_k| < \infty}\),

where \(b_k = \sum_{i=k+1}^{\infty} v_i^{-1}a_1\).

In order to prove Theorem 5, we need the following lemmas.

**Lemma 5:** (i) \((\Delta_v'(\ell_\infty))^\alpha = G_1\), where

\[G_1 = \{a = (a_k): \sum_k k |v_k^{-1}a_k| < \infty\},\]

(ii) \((\Delta_v'(\ell_\infty))^\beta = G_2\), where

\[G_2 = \{a = (a_k): \sum_k kv_k^{-1}a_k \text{ is convergent and} \sum_k |b_k| < \infty\};\]

(iii) \((\Delta_v'(\ell_\infty))^\gamma = G_3\), where

\[G_3 = \{a = (a_k): \sup_n \sum_{k=1}^n kv_k^{-1}a_k < \infty \text{ and} \sum_k |b_k| < \infty\}.

**Proof:** (i) Let \(a \in G_1\) and \(x \in \Delta_v'(\ell_\infty)\). Then

\[\sum_k |a_kx_k| = \sum_k k |v_k^{-1}a_k| |k^{-1}| v_kx_k < \infty,\]

by Lemma 2. Hence, \(a \in (\Delta_v'(\ell_\infty))^\alpha\). Now let \(a \in (\Delta_v'(\ell_\infty))^\alpha\) which leads to \(\sum_k |a_kx_k| < \infty\) for each \(x \in \Delta_v'(\ell_\infty)\). Therefore, if we choose

\[x_k = \begin{cases} 0 & \text{if } k = 1 \\ kv_k^{-1} & \text{if } k \geq 2 \end{cases}\]

then

\[|v_1^{-1}a_1| + \sum_k |a_kx_k| = \sum_k k |v_k^{-1}a_k| < \infty\]

which implies that \(a \in G_1\).

(ii) Suppose that \(a \in G_2\). If \(x \in \Delta_v'(\ell_\infty)\), then by (2.4), there exists one and only one \(y = (y_k) \in \ell_\infty\) such that

\[x_k = - v_k^{-1} \sum_{i=1}^k y_{i-1} (y_0 = 0),\]
and hence
\[ \sum_{k=1}^{n} a_k x_k = - \sum_{k=1}^{n} a_k v_k^{-1} \sum_{i=1}^{k} y_{i-1} \]
\[ = - \sum_{k=1}^{n} (b_{k-1} - b_k) \sum_{i=1}^{k} y_{i-1} \] (3.2)
\[ = - \sum_{k=1}^{n-1} b_k y_k + b_n \sum_{k=1}^{n-1} y_k. \]

Since, by Lemma 4 (ii), \( \sum_k b_k y_k \) is absolutely convergent and
\[ b_n \sum_{k=1}^{n-1} y_k \to 0 \text{ as } n \to \infty, \]
the series \( \sum_k a_k x_k \) is convergent for each \( x \in \Delta'_v(t_\infty) \), hence \( a \in (\Delta'_v(t_\infty))^3 \).

Now let \( a \in (\Delta'_v(t_\infty))^3 \), then \( \sum_k a_k x_k \) is convergent for each \( x \in \Delta'_v(t_\infty) \). If we consider the sequence \( x = (x_k) \) defined in (3.1), then the series \( \sum_k k v_k^{-1} a_k \) converges. This implies that \( \lim b_n = 0 \) as \( n \to \infty \)
(by Lemma 4(ii)). Again using (3.2) it can be shown that \( \sum_k a_k x_k = \sum_k b_k y_k \) is convergent for all \( y \in t_\infty \). So we have \( \sum_k |b_k| < \infty \) and hence \( a \in G_2 \).

(iii) It can be proved by the same way as above that \( (\Delta'_v(t_\infty))^\eta = G_3 \), using Lemma 4(i).

Lemma 6: For \( \eta = \alpha, \beta \) or \( \gamma \), we have
\[ (\Delta'_v(t_\infty))^\eta = (\Delta'_v(c))^\eta. \]

Proof: We prove for \( \eta = \alpha \) only. For \( \eta = \beta \) and \( \gamma \), the proofs are similar.

Since \( c \in t_\infty \), then \( \Delta'_v(c) \subset \Delta'_v(t_\infty) \) and hence \( (\Delta'_v(t_\infty))^\alpha \subset (\Delta'_v(c))^\alpha \).

Let \( a \in (\Delta'_v(c))^\alpha \). Then \( \sum_k |a_k x_k| < \infty \) for every \( x \in \Delta'_v(c) \).

If we consider the sequence \( x = (x_k) \) defined in (3.1), then \( (x_k) \in \Delta'_v(c) \) and hence \( \sum_k k |v_k^{-1} a_k| < \infty \) so that \( a \in (\Delta'_v(t_\infty))^\alpha \) (by Lemma 5(i)).
This completes the proof.

**Lemma 7:** For $X = c_0$ or $c$, we have

$$(\Delta'_\gamma(X))_{\gamma} = (\Delta_\gamma(X))_{\gamma}$$

where $\gamma = \alpha, \beta$ or $\gamma$.

**Proof:** We prove for $\gamma = \alpha$ and $X = c_0$ only. Since $\Delta'_\gamma(c_0) \subseteq \Delta_\gamma(c_0)$, it is clear that

$$(\Delta_\gamma(c_0))_{\alpha} \subseteq (\Delta'_\gamma(c_0))_{\alpha}.$$ 

Let $a \in (\Delta'_\gamma(c_0))_{\alpha}$, so that $\sum_k k |v_k^{-1}a_k| < \infty$. If $x \in \Delta_\gamma(c_0)$, then $\sup_k k^{-1} |v_k x_k| < \infty$ (by Lemma 2). Hence,

$$\sum_k a_k x_k = \sum_k k |v_k^{-1}a_k| k^{-1} |v_k x_k| < \infty,$$

which implies $a \in (\Delta_\gamma(c_0))_{\alpha}$.

The proofs for the other cases are similar.

Now, the proof of Theorem 5 is immediate by Lemma 5,6 and 7.

Assuming $\nu = (k)$ in Theorem 5, we obtain the following results that give us the $\alpha$, $\beta$- and $\gamma$- duals of the sequence spaces $\Delta_\gamma(c_0)$ and $\Delta_\gamma(c)$ in terms of some well-known sequence spaces.

**Corollary 3:** For $X = c_0$ or $c$ we have

(i) $(\Delta(k)(X))_{\alpha} = c_0,$

(ii) $(\Delta(k)(X))_{\beta} = \gamma \cap \Lambda(k),$

(iii) $(\Delta(k)(X))_{\gamma} = m_\infty \cap \Lambda(k),$

where $\gamma = \{a = (a_k) : \sum_k a_k \text{ converges}\}$, $m_\infty = \{a = (a_k) : \sup_n \sum_{k=1}^n a_k < \infty\}$ and $\Lambda(k) = \{a = (a_k) : \sum_k \sum_{j=k+1}^\infty j^{-1}a_j < \infty\}$.

Putting $\nu = (1,1,\ldots)$ in Theorem 5, we obtain the following results.

**Corollary 4([2]):** For $X = c_0$ or $c$ we have
(i) \( (\Delta(X))^\alpha = \{a = (a_k) : \sum_k k |a_k| < \infty\} \),

(ii) \( (\Delta(X))^\beta = \{a = (a_k) : \sum_k ka_k \text{ converges and } \sum_k |b'_k| < \infty\} \),

(iii) \( (\Delta(X))^\gamma = \{a = (a_k) : \sup_n \left| \sum_{k-1}^n ka_k \right| < \infty \text{ and } \sum_k |b'_k| < \infty\} \)

where \( b'_k = \sum_{l=k+1}^{\infty} a_l \).

Acknowledgement: I thank Prof. P.D. Srivastava, Department of Mathematics, Indian Institute of Technology, Kharagpur, for his help during this investigation.

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