SOME PROPERTIES WHICH A RING OF HOLOMORPHIC FUNCTIONS
ON A NON-EMPTY SUBSET OF AN OPEN RIEMANN SURFACE
MIGHT HAVE

N. İSPIR and İ.K. ÖZKIN

Department of Mathematics, Faculty of Sciences, Ankara University-TURKEY

ABSTRACT

Su (1972) proved that for any two subsets X, Y of C, the complex plane, X and Y are
conformally homeomorphic if there is an isomorphism between H(X) and H(Y) which is the
identity on constant functions. Minda (1976) extended the method to the rings of holomorphic
functions on any subsets of open Riemann surfaces. Royden (1963) listed some properties which
a ring of functions might have. In this paper, using the results of Su, Minda, and Royden we
present some properties of the subring Rφ.

Introduction

In this paper R and S will denote open Riemann surfaces and X,Y
will be non-empty subsets of R and S, respectively. A function
φ: X → S is said to be analytic if for each point p ∈ X there is an open
neighborhood U_p of p and an analytic function ϕ_p: U_p → S such that
ϕ_p and φ coincide on U_p ∩ X. This is equivalent to assuming that there
is a single open set U ⊃ X and an analytic function ϕ: U → S such that
ϕ|X = φ, (Minda, 1973); in this reference it is assumed that
R=S=C, but the method readily extends to the present situation. Let
A(X,Y) denote the set of all analytic functions φ: X → S with φ(X) ⊂ Y.
For Y=S=C, a function in A(X,C) is called holomorphic and we
write A(X,C) = H(X). Thus H(X) is the set of all holomorphic functions
on X which is non-empty subset of an open Riemann surface R. It is
well known that H(X) is an integral domain under pointwise addition
and multiplication. In fact, H(X) is an algebra over both the complex
numbers C and the real numbers R.

It is known that if R and S are open Riemann surfaces and X,Y
non-empty subsets of R,S respectively, and if φ is a C-algebra homo-

ISSN 02571-081 A. Ü. Basimevi
morphism of $H(Y)$ into $H(X)$ mapping each constant function onto itself, then there is a unique analytic mapping $\varphi \in \mathcal{A}(X,Y)$ such that $\Phi(g) = g\varphi$ for $g \in H(Y)$, Minda (1976). Also if $\Phi$ is an isomorphism of $H(Y)$ into $H(X)$, then $\varphi$ is a one-to-one mapping of $X$ into $Y$. Thus a subring $R^*$ of $H(X)$ is a homomorphic image of a ring $H(Y)$ under a $C$-algebra homomorphism if and only if $R^* = R_\varphi = \{g\varphi : g \in H(Y), \varphi \in \mathcal{A}(X,Y)\}$. Moreover, if $\varphi$ is a one-to-one analytic mapping of $X$ onto $Y$ and $\Phi$ maps $H(Y)$ into $H(X)$ in such a way that $\Phi(g) = g\varphi$, $g \in H(Y)$, then $\Phi(H(Y)) = H(X)$.

In this paper we are concerned with the proper subrings $R^*$ of $H(X)$ which are $C$-isomorphic images of $H(Y)$, the algebra of all holomorphic functions on a non-empty subset $Y$ of an open, Riemann surface. S. Royden (1963) listed the following properties which a ring of functions might have. He and others have shown that if $G$ is an open set in the plane or on open Riemann surface, then $H(G)$ has these properties.

(a) If $f \in H(G)$ and if $f$ is never zero, then $f$ has a multiplicative inverse in $H(G)$.

(β) If $f_1, \ldots, f_n$ are elements of $H(G)$ with no common zeros, then there are elements $e_1, \ldots, e_n$ in $H(G)$ such that $e_1f_1 + \ldots + e_nf_n = 1$.

(γ) If $f \in H(G)$ and $f$ is not identically zero, there are a finite number of functions $f_1, \ldots, f_n$ in $H(G)$ which separate the zeros of $f$.

According to Minda (1976), in case that $Y$ is a non-empty subset of an open Riemann surface $S$, $H(Y)$ the algebra of holomorphic functions on $Y$ has the following similar properties:

(a*) If $f \in H(Y)$ and $f(z) \neq 0$ for every $z \in Y$, then $\frac{1}{f} \in H(Y)$.

(β*) If $f_1, \ldots, f_n$ belong to $H(Y)$ and have no common zero, then there are functions $e_1, \ldots, e_n$ in $H(Y)$ such that $e_1f_1 + \ldots + e_nf_n = 1$.

(γ*) If $f \in H(Y)$ and $f \neq 0$, then there is a set $\{f_1, \ldots, f_n\}$ contained in $H(Y)$ such that $x \neq y$ and $f(x) = f(y) = 0$, then there is a function $f_i, i = 1, \ldots, n$, such that $f_i(x) \neq f_i(y)$. 

In the next section we shall establish whether the proper subrings of \( H(X) \) which are \( C \)-isomorphic images of \( H(Y) \) will have these properties. But before passing to the next section let us state the necessary additional properties of proper subrings of \( H(X) \).

Suppose \( X \) and \( Y \) non-empty subsets of open Reimann surfaces \( R \) and \( S \) respectively. We define a mapping of \( H(Y) \) into \( H(X) \) by \( \Phi(g) = g \circ \varphi \) for \( g \in H(Y) \). \( g \circ \varphi \) is holomorphic on \( X \) and \( \Phi \) is a \( C \)-algebra homomorphism. The image of \( \Phi \), \( R_\varphi = \Phi(H(Y)) \) is a subring of \( H(X) \). \( R_\varphi \) contains the constant functions, denoted by \( C \), since \( C \subseteq H(Y) \) and \( \Phi(\lambda) = \lambda \) for \( \lambda \in C \).

Now, if \( \Phi: H(Y) \to H(X) \) is a ring homomorphism defined by \( \Phi(g) = g \circ \varphi \) for \( g \in H(Y) \), \( \varphi \in A(X,Y) \) and if \( R_\varphi = \Phi(H(Y)) \), then the following three conditions are equivalent:

(a) \( R_\varphi \) properly contains the constant functions.

(b) \( \varphi \) is not a constant function.

(c) \( H(Y) \) is isomorphic to \( R_\varphi \).

It is clear that these are the relations between \( \Phi, \varphi, \) and \( R_\varphi \). Thus a subring \( R^* \) of \( H(X) \) is isomorphic to \( H(Y) \) under a \( C \)-algebra isomorphism, if only if \( R^* = \{g \circ \varphi: g \in H(Y), \varphi \in A(X,Y)\} \) and \( R^* \) properly contains \( C \) the constant functions on \( X \).

Moreover, if \( \varphi \) is a one-to-one analytic mapping of \( X \) into \( Y \), \( \lambda \) is a non-constant analytic mapping of \( X \) into \( Y \) but not one-to-one, \( \Phi(g) = g \circ \varphi \) and \( \Lambda(g) = g \circ \lambda \) for \( g \in H(Y) \), \( R_\varphi = (H(Y)), R_\lambda = \Lambda(H(Y)), \) then \( R_\varphi \) and \( R_\lambda \) are isomorphic but \( R_\varphi \neq R_\lambda \).

**The Function \( \varphi \) Determines Whether \( R_\varphi \) Will Have Properties (\( \alpha^* \)), (\( \beta^* \)), And (\( \gamma^* \)).**

**Theorem 1.** If \( \varphi \) is an analytic mapping of \( X \) onto \( Y \), then \( R_\varphi \) has property (\( \beta^* \)).

**Proof.** Let \( f_1, \ldots, f_n \) belong to \( R_\varphi \) and have no common zero. \( f_i = \Phi(h_i), i = 1, \ldots, n, \) where \( h_i \in H(Y) \). Suppose \( h_i(a) = 0 \) for \( a \in Y, i = 1, \ldots, n \). Since \( \varphi \) maps \( X \) onto \( Y \), there is \( z \in X \) such that \( \varphi(z) = a \). Then \( 0 = h_i(a) = h_i(\varphi(z)) = \Phi(h_i(z)) = f_i(z) \) for \( i = 1, \ldots, n \). This is a contradiction, thus \( h_1, \ldots, h_n \) can have no common zero, \( H(Y) \) has property (\( \beta^* \)) implies there are \( e_1, \ldots, e_n \) in \( H(Y) \) such that \( e_1 h_1 + \ldots + e_n h_n = 1 \).
1 = Φ(e_1 h_1 + \ldots + e_n h_n) = Φ(e_1 h_1) + \ldots + Φ(e_n h_n) \\
= Φ(e_1) Φ(h_1) + \ldots + Φ(e_n) Φ(h_n) = Φ(e_i) f_i + \ldots + Φ(e_n) f_n \\
which implies there are functions Φ(e_i), i=1,\ldots,n, in R_φ such that 
Φ(e_i) f_i + \ldots + Φ(e_n) f_n = 1.

If a set of functions has property (β*), then it has property (α*) because when n=1, (β*) is (α*). If φ is an onto mapping, R_φ has properties (α*) and (β*).

**Theorem 2.** If R_φ has property (α*) and R_φ properly contains the constant functions, then φ maps X onto Y.

**Proof.** Let a ∈ Y. Since H(S) is an algebra of all holomorphic functions on the open Riemann surface S, then there is g' ∈ H(S) such that g'(a) = 0 and g'(w) ≠ 0 for w ≠ a, Behnke and Sommer (1962). Set g' |Y = g. Then g(a) = 0 and g(w) ≠ 0 when a ≠ w, φ(g) ∈ R_φ.

If Φ(g) (z) = Φ(φ(z)) ≠ 0 for z ∈ X, then there is h ∈ R_φ such that
Φ(g) h = 1. h ∈ Φ(H(Y)) implies h = Φ(k), k ∈ H(Y). Φ(g) Φ(k) = 1 implies Φ(gk) = 1 but Φ is an isomorphism, thus gk = 1 and g(a) k(a) = 1. This contradicts g(a) = 0. Therefore g(φ(z)) = 0 for some z ∈ X and φ(z) = a for some z ∈ X. Thus φ is a mapping of X onto Y.

If R_φ has property (β*) and R_φ ≠ C, the same result holds.

**Theorem 3.** R_φ has property (Γ*) if φ is a one-to-one mapping of X into Y.

**Proof.** Let f ∈ R_φ and is f not identically zero. f = Φ(h) for some h ∈ H(Y). h is not the constant 0 or else f = Φ(h) = 0. Since H(Y) has property (Γ*), there are functions h_1,\ldots,h_n in H(Y) such that if z ≠ w and h(z) = h(w) = 0, then there is h_i such that h_1(z) ≠ h_1(w).

Φ(h_i): 1 ≤ i ≤ n} ⊆ R_φ. Suppose x ≠ y and f(x) = f(y) = 0. Then h(φ(x)) = h(φ(y)) = 0 and φ is one-to-one implies φ(x) ≠ φ(y), so there is a function h_1, i=1,\ldots,n, such that h_1(φ(x)) ≠ h_1(φ(y)) or Φ(h_1(x)) ≠ Φ(h_1(y)).

**Theorem 4.** R_φ separates the points of X if and only if φ is one-to-one.

**Proof.** Suppose R_φ separates the points of X. Let x and y belong to X, x ≠ y. Then there is a ∈ R_φ such that f(x) = f(y). f = Φ(g) = goφ implies g(φ(x)) ≠ g(φ(y)). If φ(x) = φ(y), then g(φ(x)) = g(φ(y)) for every g ∈ H(Y). Thus φ(x) ≠ φ(y) and φ is a one-to-one mapping of X into Y.
Now suppose $\varphi$ is a one-to-one mapping of $X$ into $Y$. Let $x, y$ belong to $X, x \neq y$. Then $\varphi(x), \varphi(y)$ belong to $Y$ and $\varphi(x) \neq \varphi(y)$. $H(Y)$ separates the points of $Y$ implies there is $g \in H(Y)$ such that $g(\varphi(x)) \neq g(\varphi(y))$ which implies $\Phi(g(x)) \neq \Phi(g(y))$. $\Phi(g) \in R_\varphi$. Thus $R_\varphi$ separates the points of $X$.

**Theorem 5.** If $R_\varphi$ properly contains $C$ and has property $(\Upsilon^*)$, then $\varphi$ is a one-to-one mapping of $X$ into $Y$.

**Proof.** By Theorem 3, $R_\varphi$ separates the points of $X$ if and only if $\varphi$ is a one-to-one function. We shall show $R_\varphi \neq C$ and $R_\varphi$ has property $(\Upsilon^*)$ implies $R_\varphi$ separates the points of $X$. Let $x, y$ belong to $X, x \neq y$. If there is $f \in R_\varphi, f$ is not identically zero, such that $f(x) = f(y) = 0$, then there is a set of function $\{f_1, \ldots, f_n\}$ in $R_\varphi$ and a function $f_i$ in the set such that $f_i(x) \neq f_i(y)$ by property $(\Upsilon^*)$. Since $R_\varphi \neq C$ there is $g \in R_\varphi$ such that $g$ is not a constant function. If $g(x) \neq g(y)$, then $g$ separates $x$ and $y$. If $g(x) = g(y) = c$, then since $c(x) = c$ belong to $R_\varphi, g- c$ belongs to $R_\varphi$ and $(g- c)(x) = (g- c)(y) = 0$. $g \neq c$ since $g$ is not a constant function. Since $R_\varphi$ has property $(\Upsilon^*)$ there is $h \in R_\varphi$ such that $h(x) \neq h(y)$. $R_\varphi$ separates the points of $X$ implies $\varphi$ is one-to-one.

We make the following

**Observations**

1. If $R_\varphi \neq C$ and has property $(\alpha^*)$, then $\varphi$ is an onto mapping from $X$ to $Y$. If $R_\varphi$ is to be a proper subring of $H(X)$, then $\varphi$ may not be a one-to-one mapping also. This implies $R_\varphi$ does not separate the points of $X$.

2. If $R_\varphi \neq C, R_\varphi$ can not have both properties $(\alpha^*)$ and $(\Upsilon^*)$ because then $\varphi$ would be one-to-one and onto and $R_\varphi = H(X)$.

3. $R_\varphi$ separates the points of $X$ implies $\varphi$ is one-to-one. If $R_\varphi$ properly contains $C$ and is not $H(X)$, then $\varphi$ may not be an onto mapping and $R_\varphi$ does not have property $(\alpha^*)$.

4. By Theorems 3, 4, and 5, if $R_\varphi \neq C$ the statements: $R_\varphi$ has property $(\Upsilon^*), R_\varphi$ separates the points of $X, \varphi$ is a one-to-one mapping of $X$ into $Y$ are equivalent.

5. $R_\varphi$ can not be an ideal of $H(X)$ because $1 \in R_\varphi$. 

REFERENCES


