ORE EXTENSIONS OF ZIP AND REVERSIBLE RINGS

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ABSTRACT. We investigate Ore extensions zip and reversible rings. Let \( \alpha \) be an endomorphism and \( \delta \) an \( \alpha \)-derivation of a ring \( R \). Assume that \( R \) is an \( \alpha \)-rigid ring. Then (1) \( R \) is a right zip ring if and only if the Ore extension \( R[x; \alpha, \delta] \) is a right zip ring. (2) \( R \) is a reversible ring if and only if the Ore extension \( R[x; \alpha, \delta] \) is a reversible ring.

Throughout this work all rings \( R \) are associative with identity and modules are unital right \( R \)-modules. Given a ring \( R \), the polynomial ring over \( R \) is denoted by \( R[x] \) with \( x \) its indeterminate and \( r_R(-) \left( l_R(-) \right) \) is used for the right (left) annihilator over \( R \). Faith [6] called a ring \( R \) right zip provided that if the right annihilator \( r_R(X) \) of a subset \( X \) of \( R \) is zero, \( r_R(Y) = 0 \) for a finite subset \( Y \subseteq X \); equivalently, for a left ideal \( L \) of \( R \) with \( r_R(L) = 0 \), there exists a finitely generated left ideal \( L_1 \subseteq L \) such that \( r_R(L_1) = 0 \). \( R \) is zip if it is right and left zip. The concept of zip rings initiated by Zelmanowitz [11] appeared in [2],[3],[5],[6], and references there in.

Extensions of zip rings were studied by several authors. Beachy and Blair [2] showed that if \( R \) is a commutative zip ring, then the polynomial ring \( R[x] \) over \( R \) is zip. Hong et al. [7] showed that if \( R \) is an Armendariz ring, then \( R \) is a right zip ring if and only if \( R[x] \) is a right zip ring.

According to Chon [4], a ring \( R \) is called \emph{reversible} if \( ab = 0 \) implies \( ba = 0 \) for \( a, b \in R \). Kim and Lee [9] showed that if \( R \) is an Armendariz ring, then \( R \) is a reversible ring if and only if \( R[x] \) is a reversible ring.

In this paper, we study Ore extensions of zip rings and reversible rings. In particular, we show: Let \( \alpha \) be an endomorphism and \( \delta \) an \( \alpha \)-derivation of a ring \( R \). Assume that \( R \) is an \( \alpha \)-rigid ring. Then (1) \( R \) is a right zip ring if and only if the

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Ore extension $R[x; \alpha, \delta]$ is a right zip ring. (2) $R$ is a reversible ring if and only if the Ore extension $R[x; \alpha, \delta]$ is a reversible ring.

A ring $R$ is called a reduced ring if $a^2 = 0$ in $R$ always implies $a = 0$. Recall that for a ring $R$ with a ring endomorphism $\alpha : R \to R$ and an $\alpha$-derivation $\delta : R \to R$, the Ore extension $R[x; \alpha, \delta]$ of $R$ is the ring obtained by giving the polynomial ring over $R$ with the new multiplication
\[ xr = \alpha(r)x + \delta(r) \]
for all $r \in R$. If $\delta = 0$, we write $R[x; \alpha]$ for $R[x; \alpha, 0]$ and is called an Ore extension of endomorphism type (also called a skew polynomial ring), while $R[[x; \alpha]]$ is called a skew power series ring.

**Definition 1** (Krempa [10]) Let $\alpha$ be an endomorphism of $R$. $\alpha$ is called a rigid endomorphism if $\alpha \alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring $R$ is called to be $\alpha$-rigid if there exists a rigid endomorphism $\alpha$ of $R$.

Note that $\alpha$-rigid rings are reduced rings. If $R$ is an $\alpha$-rigid ring and $r^2 = 0$ for $r \in R$, then $r\alpha(r)\alpha(r\alpha(r)) = r\alpha(r^2)\alpha^2(r) = 0$. Thus $r\alpha(r) = 0$ and so $r = 0$. Therefore, $R$ is reduced.

In this paper, we let $\alpha$ be an endomorphism of $R$ and $\delta$ an $\alpha$-derivation of $R$, unless especially noted. We need the following lemmas:

**Lemma 2** ([8, Lemma 4]) Let $R$ be an $\alpha$-rigid ring and $a, b \in R$. Then we have the following:

(i) If $ab = 0$ then $\alpha^n(b) = \alpha^n(a)b = 0$ for any positive integer $n$.

(ii) If $ab = 0$ then $a\delta^m(b) = \delta^m(a)b = 0$ for any positive integer $m$.

(iii) If $\alpha^k(b) = 0 = \alpha^k(a)b$ for some positive integer $k$, then $ab = 0$.

**Lemma 3** ([8, Proposition 6]) Suppose that $R$ is an $\alpha$-rigid ring. Let $p = \sum_{i=0}^m a_i x^i$ and $q = \sum_{j=0}^n b_j x^j$ in $R[x; \alpha, \delta]$. Then $pq = 0$ if and only if $a_i b_j = 0$ for all $0 \leq i \leq m$, $0 \leq j \leq n$.

The following theorem extends [7, Theorem 11].

**Theorem 4.** Let $R$ be an $\alpha$-rigid ring. Then $R$ is a right zip ring if and only if $R[x; \alpha, \delta]$ is a right zip ring.

**Proof.** Suppose that $R[x, \alpha, \delta]$ is right zip. Let $Y \subseteq R$ with $r_R(Y) = 0$. If $f(x) = a_0 + a_1 x + \ldots + a_n x^n \in r_R[x, \alpha, \delta](Y)$, then $bf(x) = ba_0 + ba_1 x + \ldots + ba_n x^n = 0$ for all $b \in Y$. Thus $ba_i = 0$, and so $a_i \in r_R(Y) = 0$ for all $i$. Therefore, $f(x) = 0$ and hence $r_R[x; \alpha, \delta](Y) = 0$. Since $R[x; \alpha, \delta]$ is right zip, there exists a finite subset
$Y_0 \subseteq Y$ such that $\tau_{R[x;\alpha,\delta]}(Y_0) = 0$. Thus $\tau_R(Y_0) = \tau_{R[x;\alpha,\delta]}(Y_0) \cap R = 0 \cap R = 0$. Consequently, $R$ is a right zip ring.

Conversely, assume that $R$ is a right zip ring. Let $X \subseteq R[x;\alpha,\delta]$ with $\tau_{R[x;\alpha,\delta]}(X) = 0$. Now let $Y$ be the set of all coefficients of elements in $X$. Then $Y \subseteq R$. If $a \in \tau_R(Y)$, then $ba = 0$ for all $b \in Y$. By Lemma 2, $f(x)a = 0$ for all $f(x) \in X$, and so $a \in \tau_{R[x;\alpha,\delta]}(X) = 0$. That is $\tau_R(Y) = 0$. Since $R$ is a right zip, there exists a finite subset $Y_0 \subseteq Y$ such that $\tau_R(Y_0) = 0$. For each $a \in Y_0$, there exists $h_a(x) \in X$ such that at least one of the coefficients of $h_a(x)$ is $a$. Let $X_0$ be a minimal subset of $X$ such that $h_a(x) \in X_0$ for each $a \in Y_0$. Then $X_0$ is a nonempty finite subset of $X$. Let $Y'$ be the set of all coefficients of elements in $X_0$. Then $Y_0 \subseteq Y'$ and so $\tau_R(Y') \subseteq \tau_R(Y_0) = 0$. If $f(x) = a_0 + a_1 x + \ldots + a_k x^k \in \tau_{R[x;\alpha,\delta]}(X_0)$, then $g(x)f(x) = 0$ for all $g(x) = b_0 + b_1 x + \ldots + b_t x^t \in X_0$. Since $R$ is $\alpha$-rigid, then $b_0 a_j = 0$ for all $i$ and $j$, by Lemma 3. Thus $a_j \in \tau_R(Y') = 0$ for all $j$, and so $f(x) = 0$. Hence $\tau_{R[x;\alpha,\delta]}(X_0) = 0$ and therefore $R[x;\alpha,\delta]$ is a right zip ring. □

**Corollary 5.** Let $R$ be an $\alpha$-rigid ring. Then $R$ is a right zip ring if and only if $R[x;\alpha]$ is a right zip ring.

**Corollary 6.** Let $R$ be a reduced ring. Then $R$ is a right zip ring if and only if $R[x]$ is a right zip ring.

**Theorem 7.** Let $R$ be an $\alpha$-rigid ring. Then $R$ is a right zip ring if and only if $R[[x;\alpha]]$ is a right zip ring.

**Proof.** Similar to the proof of Theorem 4 by using Lemma 2 and [8, Proposition 17]. □

**Corollary 8.** Let $R$ be a reduced ring. Then $R$ is a right zip ring if and only if $R[[x]]$ is a right zip ring.

**Example 9.** Let $R = \mathbb{Z}_2[y]/(y^2)$, where $(y^2)$ is a principal ideal generated by $y^2$ of the polynomial ring $\mathbb{Z}_2[y]$. Since $R$ is finite and commutative, $R$ is a zip ring. Now, let $\alpha$ be the identity map on $R$ and we define an $\alpha$-derivation $\delta$ on $R$ by $\delta(y + (y^2)) = 1 + (y^2)$. Then $R$ is not $\alpha$-rigid since $R$ is not reduced. However, by [1, Example 11] we get

$$R[x;\alpha,\delta] = R[x;\delta] \cong \text{Mat}_2(\mathbb{Z}_2[y^2]) \cong \text{Mat}_2(\mathbb{Z}_2[t]).$$

Since $\mathbb{Z}_2$ is Armendariz and zip rings, $\mathbb{Z}_2[t]$ is a zip ring by [7, Theorem 11] and so $\text{Mat}_2(\mathbb{Z}_2[t])$ is a zip ring by [3, Proposition 1]. Therefore $R[x;\alpha,\delta]$ is a zip ring.
In the following we obtain more examples of zip rings. Let $R$ be an algebra over a commutative ring $S$. Recall that the Dorroh extension of $R$ by $S$ is the ring $R \times S = D(R, S)$ with operations

$$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$$

$$(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$$

where $r_1, r_2 \in R$ and $s_1, s_2 \in S$. Let $R$ be a commutative ring, $M$ be an $R$-module and $\sigma$ be an endomorphism of $R$. Give $R \oplus M = N(R, M)$ a (possibly noncommutative) ring structure with multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, \sigma(r_1)m_2 + r_2m_1)$$

where $r_1, r_2 \in R$ and $m_1, m_2 \in M$. We shall call this extension the Nagata extension of $R$ by $M$ and $\sigma$.

**Proposition 10.** Let $R$ be a commutative zip ring. Then the Nagata extension of $R$ by $R$ is a left zip ring.

**Proof.** Assume that $R$ is a zip ring and $X \subseteq N(R, R)$ with $l_{N(R, R)}(X) = 0$. Let $Y = \{x \in R \mid (x, y) \in X\} \subseteq R$. If $b \in r_R(Y)$ then $(0, b)(x, y) = (0x, \sigma(0)y + xb) = (0, 0)$ for any $(x, y) \in X$. Thus $(0, b) \in l_{N(R, R)}(X) = 0$ and so $b = 0$. Therefore $r_R(Y) = 0$. Since $R$ is a right zip, there exists a finite subset $Y_0 = \{x_1, x_2, \ldots, x_m\} \subseteq Y$ such that $r_R(Y_0) = 0$. Let $X_0 = \{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m) \mid (x_i, y_i) \in X, 1 \leq i \leq m\} \subseteq X$. If $(a, b) \in l_{N(R, R)}(X_0)$ then $(a, b)(x_i, y_i) = (0, 0)$ for all $(x_i, y_i) \in X_0$. Thus $(0, 0) = (a, b)(x_i, y_i) = (ax_i, \sigma(a)y_i + xb)$. So $ax_i = 0$ and $\sigma(a)y_i + xb = 0$. Then $a \in l_R(Y_0) = r_R(Y_0) = 0$ since $R$ is commutative. Hence $\sigma(a)y_i + xb = x_i b = 0$ and so $b \in r_R(Y_0) = 0$. Consequently $l_{N(R, R)}(X_0) = 0$ and therefore $N(R, R)$ is a left zip ring. \[\square\]

**Proposition 11.** Let $R$ be a commutative ring with $2^{-1} \in R$. If the Dorroh extension of $R$ by $R$ is a right zip ring then $R$ is also right zip ring.

**Proof.** Assume $D(R, R)$ is a right zip ring and $X \subseteq R$ with $r_R(X) = 0$. Let $Y = \{(x, x) \mid x \in X\} \subseteq D(R, R)$. If $(a, b) \in r_{D(R, R)}(Y)$, then $(x, x)(a, b) = (0, 0)$ for all $x \in X$. Thus $(xa + xa + bx, xb) = (0, 0)$. So $2xa + bx = 0$ and $xb = 0$. Thus $b \in r_R(X) = 0$ and hence $b = 0$. Therefore $2xa = 0$. By hypothesis $xa = 0$ and so $a \in r_R(X) = 0$ and hence $a = 0$. Consequently, $r_{D(R, R)}(Y) = 0$. Since $D(R, R)$ is a right zip ring, there exists a finite subset $Y_0 = \{(x_1, x_1), (x_2, x_2), \ldots, (x_m, x_m)\} \subseteq Y$ such that $r_{D(R, R)}(Y_0) = 0$. Let $X_0 = \{x_1, x_2, \ldots, x_m\} \subseteq X$. If $c \in r_R(X_0)$, then $(x_i, x_i)(c, 0) = (x_ic + x_ic + 0x_i, x_i0) = (0, 0)$ for $1 \leq i \leq m$. Thus $(c, 0) \in$
\( r_{D(R,R)}(Y_0) = 0 \) and so \( c = 0 \). Therefore \( r_R(X_0) = 0 \) and so \( R \) is a right zip ring.

Lemma 12. Let \( R \) be a reversible ring. Then \( R \) is a right zip ring if and only if \( R \) is a left zip ring.

Proof. Clear.

Lemma 13. If \( R \) is reduced and \( R[x; \alpha] \) is reversible then \( R \) is an \( \alpha \)-rigid ring.

Proof. If \( a\alpha(a) = 0 \) for \( a \in R \), then \( (ax)(a + ax) = a\alpha(a)x + a\alpha(a)x^2 = 0 \). Since \( R[x; \alpha] \) is reversible \( 0 = (a + ax)(ax) = a^2x + a\alpha(a)x^2 \) and so \( a^2 = 0 \). Thus \( a = 0 \) since \( R \) is reduced. Consequently \( R \) is an \( \alpha \)-rigid ring.

For any ring \( R \), if \( R[x; \alpha, \delta] \) is a reversible ring then \( R \) is also reversible. One may conjecture that if a ring \( R \) is reversible then \( R[x; \alpha, \delta] \) is also reversible. However there may be a counterexample for this as follows.

Example 14. Let \( R = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \) and define \( \alpha : R \to R \) by \( \alpha((a,b)) = (b,a) \) for \( a, b \in \mathbb{Z}_3 \), where \( \mathbb{Z}_3 \) is the ring of integers modulo 3. Then \( \alpha \) is an automorphism of \( R \). Thus \( R \) is reversible and \( R \) is not \( \alpha \)-rigid. Take \( f(x) = (1,0) + (0,1)x \) and \( g(x) = (0,1) + (0,1)x \) in \( R[x; \alpha] \). Then
\[
 f(x)g(x) = (1,0)(0,1) + \underbrace{(1,0)(0,1) + (0,1)(1,0)x + (0,1)(1,0)x^2} = (0,0)
\]
but
\[
 g(x)f(x) = (0,1)(1,0) + \underbrace{(0,1)(0,1) + (0,1)(0,1)x + (0,1)(1,0)x^2} = (0,1)x \neq 0
\]
Hence \( R[x; \alpha] \) is not reversible.

Moreover, this example shows that the condition "\( R \) is \( \alpha \)-rigid" in the following theorem is not superfluous.

Theorem 15. Let \( R \) be an \( \alpha \)-rigid ring. Then \( R \) is a reversible ring if and only if \( R[x; \alpha, \delta] \) is a reversible ring.

Proof. Assume \( R[x; \alpha, \delta] \) is a reversible ring. Since class of reversible rings is closed under subrings, \( R \) is a reversible ring.

Conversely, assume that \( R \) is a reversible ring. Since \( R \) is \( \alpha \)-rigid, \( R[x; \alpha, \delta] \) is a reduced ring by [8, Proposition 5]. Therefore \( R[x; \alpha, \delta] \) is a reversible ring.
Theorem 16. Let $R$ be an $\alpha$-rigid ring. Then $R$ is a reversible ring if and only if $R[[x; \alpha]]$ is a reversible ring.

Proof. Proof is clear by [8, Corollary 18].

ÖZET: Bu makalede zip ve reversible halkaların Ore genişlemeleri çalışılmıştır. $R$ bir halka, $\alpha$; $R$ nin bir endomorfizması ve $\delta$ bir $\alpha$-türev olmak üzere; aşağıdaki ispatlanmıştır. (1) $R$ nin bir sağ zip halka olması için gerek ve yeten koşul $R[x; \alpha, \delta]$ nin bir sağ zip halka olmasını. (2) $R$ nin bir reversible halka olması için gerek ve yeten koşul $R[x; \alpha, \delta]$ nin bir reversible halka olmasını.

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