ON THE CLASSIFICATION OF FUZZY PROJECTIVE LINES OF FUZZY 3-DIMENSIONAL PROJECTIVE SPACE

Z. AKÇA, A. BAYAR AND S. EKMEKÇİ

ABSTRACT. In this work, the classifications of fuzzy vector planes of fuzzy 4-dimensional vector space and fuzzy projective lines of fuzzy 3-dimensional projective space from fuzzy 4-dimensional vector space are given.

1. INTRODUCTION AND PRELIMINARIES

A general definition of a fuzzy $n$-dimensional projective space $\lambda$ which is obtained from fuzzy $(n + 1)$-dimensional vector space $V$ over some field $K$ and a method to find a fuzzy projective line and a fuzzy projective plane are given in [3]. Firstly, the classification of fuzzy vector planes of fuzzy 4-dimensional vector space are introduced. And then we give the classification of the fuzzy projective lines of fuzzy 3-dimensional projective space, from fuzzy 4-dimensional vector space.

The following definitions and theorems concerning the basic concepts of the subject has been taken from [3] with some small modifications.

Definition 1.1. Let $\lambda : V \to [0, 1]$ be a fuzzy set on $V$. Then we call $\lambda$ a fuzzy vector space on $V$ if and only if $\lambda(a.\overline{u} + b.\overline{v}) \geq \lambda(\overline{u}) \land \lambda(\overline{v})$, $\forall \overline{u}, \overline{v} \in V$ and $\forall a, b \in K$.

Proposition 1. Let $V$ be a vector space over some field $K$, $\overline{u}, \overline{v} \in V$ and $a \in K \setminus \{0\}$. If $\lambda : V \to [0, 1]$ is a fuzzy vector space, then we have:

(i) $\lambda(a.\overline{u}) = \lambda(\overline{u})$;
(ii) $\lambda(\overline{v}) = \sup_{\overline{u} \in V} \lambda(\overline{u})$;
(iii) if $\lambda(\overline{u}) \neq \lambda(\overline{v})$, we have $\lambda(\overline{u} + \overline{v}) = \lambda(\overline{u}) \land \lambda(\overline{v})$.

Definition 1.2. Let $\lambda$ is a fuzzy vector space on $V$. The subspace $L$, (linearly) generated by $\text{Supp}(\lambda)$ (supp$(\lambda) = \{x \in V : \lambda(x) = 0\}$, is called the base vector space of $\lambda$. The dimension $d(\lambda)$ of a fuzzy vector space of $V$ is the dimension of its base subspace.

Received by the editors April 29, 2006; Rev. Sept. 25, 2006; Accepted: Nov.14, 2006.

Key words and phrases. Fuzzy projective space, Fuzzy point, Fuzzy vector line, Fuzzy vector plane, Fuzzy projective line.

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Definition 1.3. If $U$ is an $i$–dimensional subspace of $V$, and $(\lambda, U)$ is a fuzzy vector space, then it is called a fuzzy $i$–dimensional vector space on $U$. If $i = 1$, i.e. $U$ is a vector line, then $(\lambda, U)$ is a fuzzy vector line on $U$, if $i = 2$, i.e. $U$ is a plane, $(\lambda, U)$ will be called a fuzzy vector plane on $U$. If $i = n - 1$, then $(\lambda, U)$ is called a fuzzy vector hyperplane on $U$.

Let $V$ be an $n$–dimensional vector space over some field $K$, with $n \geq 2$. Let $L$ be a vector line of $V$, so $L$ is uniquely defined by some nonzero vector $\bar{u}$. Let $\alpha$ be a vector plane of the $n$–dimensional vector space $V$ ($n \geq 3$), then we know that $\alpha$ is uniquely defined by two linearly independent vectors $\bar{u}$ and $\bar{v}$.

Theorem 1.4. If $\lambda : L \rightarrow [0, 1]$ is a fuzzy vector line on $L$, then $\lambda(\bar{u}) = \lambda(\bar{v})$, $\forall \bar{u}, \bar{v} \in L \setminus \{\bar{0}\}$, and $\lambda(\bar{0}) \geq \lambda(\bar{u})$, $\forall \bar{u} \in L$.

Theorem 1.5. If $\lambda : \alpha \rightarrow [0, 1]$ is a fuzzy vector plane on $\alpha$, then there exists a vector line $L$ of $\alpha$ and real numbers $a_0 \geq a_1 \geq a_2 \in [0, 1]$ such that $\lambda$ is of the following form:

$$
\begin{align*}
\lambda : & \quad \alpha \rightarrow [0, 1] \\
\bar{0} & \rightarrow a_0 \\
\bar{u} & \rightarrow a_1 \text{ for } \bar{u} \in L \setminus \{\bar{0}\} \\
\bar{u} & \rightarrow a_2 \text{ for } \bar{u} \in \alpha \setminus L,
\end{align*}
$$

Definition 1.6. Suppose $V$ is an $n$–dimensional vector space. A flag in $V$ is a sequence of distinct, non-trivial subspaces $(U_0, U_1, \ldots, U_m)$ such that $U_j \subset U_i$ for all $j < i < n - 1$. The rank of a flag is the number of subspaces it contains. A maximal flag in $V$ is a flag of length $n$.

2. Fuzzy Vector Planes of Fuzzy 4-Dimensional Vector Spaces

In this work, now we classify fuzzy 2-dimensional subspaces of fuzzy 4-dimensional vector spaces to classify fuzzy projective lines of fuzzy 3-dimensional projective space. Since a subspace should not necessarily have the same values (membership degrees different from $a_0$) in its points as the whole space [3], this classification is given in the following theorem.

Theorem 2.1. Let $V$ be a 4-dimensional vector space over some field $K$ and $\lambda : V \rightarrow [0, 1]$ be a fuzzy vector space on $V$. Then the fuzzy 4-dimensional vector space $\lambda$ has exactly six kinds of fuzzy vector planes.

Proof. Let $\lambda : V \rightarrow [0, 1]$ is a fuzzy vector space on $V$ and $(U_0, U_1, U_2, U_3, V)$ is a maximal flag, then there exists a vector plane $\alpha$ of $V$ and a base line $L$ of $\alpha$ and real numbers $a_0 \geq a \geq b \geq c \geq d \in [0, 1]$ such that
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\[ \lambda : \ V \rightarrow [0, 1] \]
\[ \overline{a} \rightarrow a_0 \]
\[ \overline{a} \rightarrow a \text{ for } \overline{u} \in U_1 \setminus \{U_0\} \]
\[ \overline{a} \rightarrow b \text{ for } \overline{u} \in U_2 \setminus U_1 \]
\[ \overline{a} \rightarrow c \text{ for } \overline{u} \in U_3 \setminus U_2 \]
\[ \overline{a} \rightarrow d \text{ for } \overline{u} \in V \setminus U_3. \]

The number of points different from zero on a vector line is denoted by \( p, \ q \) counts the number of vector lines \( L_j \) passing through zero point different from base line in \( U_2, \ r \) counts the number of vector lines \( L_k \) passing through zero point different from base line in \( U_3 \setminus L_j \) and \( s \) counts the number of vector lines \( L_t \) in \( V \setminus U_3 \). These fuzzy vector planes of \( \lambda \) are of one of the following forms:

1) Let \( \alpha_j \) be 2-dimensional vector spaces,

\[ \lambda_{ij} : \ \alpha_j \rightarrow [0, 1] \]
\[ \overline{a} \rightarrow a_0 \]
\[ \overline{a} \rightarrow a_i \text{ for } \overline{u} \in L \setminus \{\overline{a}\} \]
\[ \overline{a} \rightarrow b_{ij} \text{ for } \overline{u} \in \alpha_j \setminus L \]

such that \( a_i \geq b_{ij}, \ i \in \{1, \ldots, p\}, \ j \in \{1, \ldots, q\}. \)

2) Let \( \beta_k \) be 2-dimensional vector spaces, which

\[ \mu_{ik} : \ \beta_k \rightarrow [0, 1] \]
\[ \overline{a} \rightarrow a_0 \]
\[ \overline{a} \rightarrow a_i \text{ for } \overline{u} \in L \setminus \{\overline{a}\} \]
\[ \overline{a} \rightarrow c_{ik} \text{ for } \overline{u} \in \beta_k \setminus L \]

such that \( a_i \geq c_{ik}, \ i \in \{1, \ldots, p\}, \ k \in \{1, \ldots, r\}. \)

3) Let \( \gamma_t \) be 2-dimensional vector spaces,

\[ \delta_{it} : \ \gamma_t \rightarrow [0, 1] \]
\[ \overline{a} \rightarrow a_0 \]
\[ \overline{a} \rightarrow a_i \text{ for } \overline{u} \in L \setminus \{\overline{a}\} \]
\[ \overline{a} \rightarrow d_{it} \text{ for } \overline{u} \in \gamma_t \setminus L \]

such that \( a_i \geq d_{it}, \ i \in \{1, \ldots, p\}, \ t \in \{1, \ldots, s\}. \)

4) Let \( \alpha_{jk} \) be 2-dimensional vector spaces,

\[ \psi_{ijk} : \ \alpha_{jk} \rightarrow [0, 1] \]
\[ \overline{a} \rightarrow a_0 \]
\[ \overline{a} \rightarrow b_{ij} \text{ for } \overline{u} \in L_j \setminus \{\overline{a}\} \]
\[ \overline{a} \rightarrow c_{ik} \text{ for } \overline{u} \in \alpha_{jk} \setminus L_j \]

such that \( b_{ij} \geq c_{ik}, \ i \in \{1, \ldots, p\}, \ j \in \{1, \ldots, q\}, \ k \in \{1, \ldots, r\}. \)

5) Let \( \beta_{jt} \) be 2-dimensional vector spaces,
\( \varphi_{ijt} : \beta_{jt} \rightarrow [0, 1] \)
\[ \begin{align*}
\bar{a} & \rightarrow a_0 \\
\bar{u} & \rightarrow b_{ij} \text{ for } \bar{u} \in L_j \setminus \{\bar{a}\} \\
\bar{u} & \rightarrow d_{jt} \text{ for } \bar{u} \in \beta_{jt} \setminus L_j
\end{align*} \]

such that \( b_{ij} \geq d_{jt}, i \in \{1, \ldots, p\}, j \in \{1, \ldots, q\}, t \in \{1, \ldots, s\} \).

6) Let \( \gamma_{kt} \) be 2-dimensional vector spaces,
\[ \eta_{ikt} : \gamma_{kt} \rightarrow [0, 1] \]
\[ \begin{align*}
\bar{a} & \rightarrow a_0 \\
\bar{u} & \rightarrow c_{ik} \text{ for } \bar{u} \in L_j \setminus \{\bar{a}\} \\
\bar{u} & \rightarrow d_{it} \text{ for } \bar{u} \in \gamma_{kt} \setminus L_j
\end{align*} \]

such that \( c_{ik} \geq d_{it}, i \in \{1, \ldots, p\}, k \in \{1, \ldots, r\}, t \in \{1, \ldots, s\} \). Any fuzzy vector plane is in the one of the six classes \( \square \).

Now, we give an example of two subclasses of fuzzy vector planes from \( \lambda_{ij} \) and \( \eta_{ikt} \).

**Example 2.2.** For \( j = 2, k = 2 \) and \( t = 3 \), fuzzy subspaces \( \lambda_{i2} \) and \( \eta_{i23} \) are given as follows:

\[ \lambda_{i2} : \alpha_2 \rightarrow [0, 1] \]
\[ \begin{align*}
\bar{a} & \rightarrow a_0 \\
\bar{u} & \rightarrow a_i \text{ for } \bar{u} \in L \setminus \{0\} \\
\bar{u} & \rightarrow b_{i2} \text{ for } \bar{u} \in \alpha_j \setminus L
\end{align*} \]

such that \( a_i \geq b_{i2} \geq i \in \{1, \ldots, p\} \) and

\[ \eta_{i23} : \gamma_{23} \rightarrow [0, 1] \]
\[ \begin{align*}
\bar{a} & \rightarrow a_0 \\
\bar{u} & \rightarrow c_{i2} \text{ for all } \bar{u} \in \beta_2 \setminus \{0\} \\
\bar{u} & \rightarrow d_{i3} \text{ for all } \bar{u} \in \gamma_{23} \setminus \{0\}
\end{align*} \]

such that \( c_{i2} \geq d_{i3}, i \in \{1, \ldots, p\} \)

3. **Fuzzy Projective Lines of Fuzzy 3-Dimensional Projective Space**

A general definition of fuzzy \( n \)-dimensional projective space \( \lambda' \) is well-known [3]. Here, we restrict ourselves to the case a fuzzy 3-dimensional projective space \( \lambda' \) from a fuzzy 4-dimensional vector space \( (\lambda, V) \), having following form:

\[ \lambda : V \rightarrow [0, 1] \]
\[ \begin{align*}
\bar{a} & \rightarrow a_0 \\
\bar{u} & \rightarrow a \text{ for } \bar{u} \in U_1 \setminus \{U_0\} \\
\bar{u} & \rightarrow b \text{ for } \bar{u} \in U_2 \setminus U_1 \\
\bar{u} & \rightarrow c \text{ for } \bar{u} \in U_3 \setminus U_2 \\
\bar{u} & \rightarrow d \text{ for } \bar{u} \in V \setminus U_3
\end{align*} \]
with $U_i$ an $i$-dimensional subspace of $V$, containing all $U_j$ for $j < i$, and $a_0 \geq a \geq b \geq c \geq d$ are reals in $[0, 1]$. We define a fuzzy 3-dimensional projective space $\lambda'$ on $V'$ as follows, where it will be denoted $FPG(3, K)$.

\[
\lambda' : V' \to [0, 1] \\
q \to a \\
p \to b \text{ for } p \in U'_1 \setminus \{q\} \\
p \to c \text{ for } p \in U'_2 \setminus U'_1 \\
p \to d \text{ for } p \in V' \setminus U'_2
\] (2)

with $q$ the fuzzy projective point corresponding to the fuzzy vector line $U_1$ in (2) and $U'_i$ the $i$-dimensional projective space, corresponding to the vector space $U_{i+1}$. Then, the sequence $(q, U'_1, U'_2, V')$ is a maximal flag and $a \geq b \geq c \geq d$ are reals in $[0, 1]$.

The following theorem deals with the classification of fuzzy projective lines of fuzzy 3-dimensional projective space from fuzzy 4-dimensional vector space.

**Theorem 3.1.** Fuzzy 3-dimensional projective space $\lambda'$ from fuzzy 4-dimensional vector space $\lambda$ over some field $K$ has exactly six kinds of fuzzy projective lines.

**Proof.** Let $\lambda'$ be fuzzy 3-dimensional projective space on $V'$. Then it is form as follows

\[
\lambda' : V' \to [0, 1] \\
q \to a \\
p \to b \text{ for } p \in U'_1 \setminus \{q\} \\
p \to c \text{ for } p \in U'_2 \setminus U'_1 \\
p \to d \text{ for } p \in V' \setminus U'_2.
\]

The fuzzy projective lines of $\lambda'$ are one of the following forms:

1) Let $L_j$ be projective lines corresponding to the vector planes $\alpha_j$, and $q$ be the projective point corresponding to the vector line $L \subseteq \alpha$.

\[
\lambda_{ij} : L_j \to [0, 1] \\
q \to a_i \\
p \to b_{ij}, \text{ for } p \in L_j \setminus \{q\}
\]
such that $a_i \geq b_{ij}$.

2) Let $M_k$ be projective lines corresponding to the vector planes $\beta_k$, and $q$ be a projective point corresponding to the vector line $L \subseteq \beta_k$.

\[
\mu_{ik} : M_k \to [0, 1] \\
q \to a_i \\
p \to c_{ik} \text{ for } p \in M_k \setminus \{q\}
\]
such that $a_i \geq c_{ik}$. 
3) Let $N_t$ be projective lines corresponding to the vector planes $\gamma_t$, and $q$ be a projective point corresponding to the vector line $L \subseteq \gamma_t$.

\[
\eta_t' : \ N_t \to [0, 1] \\
q \to a_i \\
p \to d_{it} \text{ for } p \in N_t \{q\}.
\]

such that $a_i \geq d_{it}$.

4) Let $L_{jk}$ be projective lines corresponding to the vector planes $\alpha_{jk}$.

\[
\psi_{ijk}' : \ L_{jk} \to [0, 1] \\
q_j \to b_{ij}, \text{ for } q_j \in L \\
p \to c_{ik}, \text{ for } p \in L_{jk} \{q_j\}.
\]

such that $b_{ij} \geq c_{ik}$.

5) Let $M_{jt}$ be projective lines corresponding to the vector planes $\beta_{jt}$.

\[
\varphi_{ijt}' : \ M_{jt} \to [0, 1] \\
q_j \to b_{ij}, \text{ for } q_j \in L \\
p \to d_{it}, \text{ for } p \in M_{jt} \{q_j\}.
\]

such that $b_{ij} \geq d_{it}$.

6) Let $N_{kt}$ be projective lines corresponding to the vector planes $\gamma_{kt}$.

\[
\eta_{ikt}' : \ N_{kt} \to [0, 1] \\
p \to c_{ik}, \text{ for } p \in L_j \\
p \to d_{it}, \text{ for } p \in N_{kt} \{L_j\}.
\]

such that $c_{ik} \geq d_{it}$.

One can easily see that any fuzzy projective line is in one of above the six classes.

Example 3.2. If we consider the subclasses $\lambda_{i2}$ and $\eta_{i23}$ in the example 2.1, then the subclasses of fuzzy projective lines $\lambda_{i2}'$ and $\eta_{i23}'$ from fuzzy vector planes $\lambda_{i2}$ and $\eta_{i23}$ are as follows:

\[
\lambda_{i2}' : \ L_2 \to [0, 1] \\
q \to a_i \\
p \to b_{i2} \text{ for } p \in L_2 \{q\}
\]

and

\[
\eta_{i23}' : \ N_{23} \to [0, 1] \\
p \to c_{i2} \text{ for all } p \in L_2 \\
p \to d_{i3} \text{ for all } p \in N_{23} \{L_2\}
\]

ÖZET Bu çalışmada, fuzzy 4—boyutlu vektör uzayının fuzzy vektör düzlemlerinin sınıflaması ve fuzzy 4—boyutlu vektör uzayından elde edilen fuzzy 3—boyutlu projektif uzayın fuzzy projektif doğrularının sınıflaması veriliyor.
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Current address: Eskişehir Osmangazi University, Department of Mathematics, 26480 Eskişehir, Turkey.
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