ON HARMONIC CURVATURES OF CURVES IN LORENTZIAN N-SPACE

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Abstract. In this study we consider the harmonic curvatures of a Frenet curve in Lorentzian space \( \mathbb{L}^n \). We give a characterization of the \( r^{th} \)-curvature center \( C_r(t) \) of a Frenet curve of \( \mathbb{L}^n \) with respect to its harmonic curvatures \( H_j \), \( 1 \leq j \leq r \).

1. Introduction

Let \( X = (x_1, x_2, \ldots, x_n) \) and \( Y = (y_1, y_2, \ldots, y_n) \) be nonzero vectors in \( n \)-dimensional real vector space \( \mathbb{R}^n \). For \( X, Y \in \mathbb{R}^n \)

\[
(X, Y) = -x_1y_1 + \sum_{i=2}^{n} x_i y_i
\]  

(1.1)

is called Lorentzian inner product. The couple \( \{ \mathbb{R}^n, \langle \cdot, \cdot \rangle \} \) is called Lorentzian space and denoted by \( \mathbb{L}^n \) [2]. The vector \( X \) of \( \mathbb{L}^n \) is called (see [7])

i) time-like if \( \langle X, X \rangle < 0 \),

ii) space-like if \( \langle X, X \rangle > 0 \) or \( X = 0 \),

iii) null or null vector if \( \langle X, X \rangle = 0 \), \( X \neq 0 \).

In [4] the first author consider the curvature center of the curves on a hypersurface \( M^n \) in \( \mathbb{E}^{n+1} \) which was partially contained some results from [5].

In [6] the same author consider the curvature center of the curves a hypersurface in a \( n \)-dimensional Lorentzian space \( \mathbb{L}^n \). She has shown that the locus of centers of spheres that has \( \alpha(t) \) as the \( r \)-multiple contact point with \( \alpha \) are on the \( (n - r) \)-curvature hyperplane \( D_{(n-r)}(t) \).

In this study we consider the harmonic curvatures of a Frenet curve in Lorentzian space \( \mathbb{L}^n \). We give a characterization of the \( r^{th} \)-curvature center \( C_r(t) \) of a Frenet curve of \( \mathbb{L}^n \) with respect to its harmonic curvatures \( H_j \), \( 1 \leq j \leq r \). We also consider the general helices in \( \mathbb{L}^n \).

2. Basic Definitions

The Frenet curvature and Frenet equations of curve \( \mathbb{L}^n \) can be defined as follows.
Definition 2.1. Let $\alpha : I \rightarrow \mathbb{L}^n$ be a curve in $\mathbb{L}^n$ and $k_1, k_2, \ldots, k_{(n-1)}$ the Frenet curvatures of $\alpha$. Then for the unit tangent vector $V_1 = \alpha'(t)$ over $M$ the $i^{th}$ e-curvature function $m_i$ is defined by (see [5])

$$m_i = \begin{cases} 
0, & i = 1 \\
\frac{d}{dt}(m_{i-1}) + \varepsilon_{i-2}k_{i-2}m_{i-2} & 2 < i \leq n
\end{cases}$$

where $\varepsilon_i = \langle V_i, V_i \rangle = \pm 1$.

Let $\alpha : I \rightarrow \mathbb{L}^n$ be a unit speed non-null curve in $\mathbb{L}^n$. The curve $\alpha$ is called a Frenet curve of osculating order $d$ if its higher order derivatives $\alpha'(t), \alpha''(t), \ldots, \alpha^d(t)$ are linearly independent and $\alpha'(t), \alpha''(t), \ldots, \alpha^d(t), \alpha^{d+1}(t)$ are no longer linearly independent for all $t \in I$. For each Frenet curve of order $d$ one can associate an orthonormal $d-$frame $V_1, V_2, \ldots, V_d$ along $\alpha$ (such that $\alpha'(t) = V_1$) called the Frenet frame and $d-1$ functions $k_1, k_2, \ldots, k_{(d-1)} : I \rightarrow \mathbb{B}$ called the Frenet curvatures, such that the Frenet formulas are defined in the usual way:

$$V'_1 = \nabla_{v_1} \alpha' = \varepsilon_2k_1V_2,$$
$$V'_2 = \nabla_{v_1} V_2 = -\varepsilon_1k_1V_1 + \varepsilon_3k_2V_3,$$
$$\vdots$$
$$V'_i = \nabla_{v_1} V_i = -\varepsilon_{(i-1)}k_{(i-1)}V_{(i-1)} + \varepsilon_{(i+1)}k_iV_{(i+1)},$$
$$V'_{i+1} = \nabla_{v_1} V_{i+1} = -\varepsilon_i k_i V_i$$

where $\nabla$ is the Levi-Civita connection of $\mathbb{L}^n$.

A non-null curve $\alpha : I \rightarrow \mathbb{L}^n$ is called a W-curve (or helix) of rank $d$, if $\alpha$ is a Frenet curve of osculating order $d$ and the Frenet curvatures $k_i, 1 \leq i \leq d-1$ are non-zero constants.

Definition 2.2. Let $\alpha$ be a non-null curve of osculating order $d$. The functions $H_j : I \rightarrow \mathbb{L}^n, 1 \leq j \leq d-2$, defined by

$$H_0 = 0, \quad H_1 = \frac{k_1}{k_2}$$
$$H_j = \left\{ \nabla_{v_1}(H_{j-1}) + \varepsilon_{j-2}H_{j-2}k_j \right\} \frac{\varepsilon_j}{k_{j+1}}, 2 \leq j \leq d-2$$

are called the harmonic curvatures of $\alpha$, where $k_1, k_2, \ldots, k_{(d-1)}$ are not necessarily constant [1].

Proposition 1. [8] Let $\alpha$ be a non-null curve of osculating order $d$ then

$$k_r(t) = \frac{\varepsilon_{r-2} \left( \sum_{i=1}^{r-2} H_i^2 \right)'}{2H_{r-2}H_{r-1}}, 2 \leq r \leq d-2$$

where $(H_i)'$ stands for differentiation with respect to parameter $t$. 
3. General Helices in $L^n$

In the present section we will consider general helices in $L^n$.

**Definition 3.1.** Let $\alpha$ be a non-null curve of osculating order $d$. Then $\alpha$ is called a *general helix of rank* $(d - 2)$ if (see [1])

$$\sum_{i=1}^{d-2} H_i^2 = c,$$  \hspace{1cm} (3.1)

holds, where $c$ is a real constant.

We have the following result.

**Theorem 3.2.** For the non-null curve of $L^n$, if the $j^{th}$ harmonic curvature $H_j$ ($j \neq 1$) vanishes identically, i.e. $H_j = 0$, then $\alpha$ is a general helix of rank $j - 1$.

*Proof.* Let $\alpha$ be a non-null curve of $L^n$ then by (2.3) the harmonic curvatures of $\alpha$ become

$$H_j = ((H_{j-1})' + \epsilon_{j-2}H_{j-2}k_j) \frac{\epsilon_j}{k_{j+1}}.$$  

Suppose, $H_j = 0$ then we have

$$(H_{j-1})' + \epsilon_{j-2}H_{j-2}k_j = 0.$$  \hspace{1cm} (3.2)

So, substituting (2.4) into (3.2) we may get

$$H_{j-1}H_{j-1} + H_1H_1' + H_2H_2' + \cdots + H_{j-2}H_{j-2}' = 0.$$  \hspace{1cm} (3.3)

Hence, by virtue of (3.3) an easy calculation gives

$$\sum_{i=1}^{j-1} H_i^2 = c$$

where $c = \text{const}$. This completes the proof of the theorem. $\square$

**Corollary 1.** If $H_1 = 0$ then $\alpha$ is a straight line.

**Corollary 2.** If $H_1$ is constant then $\alpha$ is a general helix of rank 1.

**Corollary 3.** If $\alpha$ is a general helix of rank 2 then $H_2' + \epsilon_1H_1k_3 = 0$.

4. Curvature Centers of a Frenet Curve

In the present part we give a characterization of the curvature centers $C_r(t)$ of a Frenet curve with respect to its harmonic curvatures $H_j$.

**Definition 4.1.** Let $\alpha : I \rightarrow \mathbb{R}^n_1 = L^n$ be a non-null curve. If $m_1, ..., m_n$ denote the $j^{th}$ e-curvature functions of $\alpha$ and $\{V_1, ..., V_n\}$ the Frenet frame field of $\alpha$ then

$$C_r(t) = \left( \alpha + \sum_{j=2}^{r} a_jm_jV_j \right)(t), \hspace{0.5cm} a_j = \pm 1$$  \hspace{1cm} (4.1)

is called $r^{th}$ $(a_1, ..., a_r)$-curvature center of $\alpha$ at the point $\alpha(t)$ [6].

By the use of (2.1) and (2.2) with (4.1) we obtain the following result.
Theorem 4.2. Let $\alpha$ be a non-null curve of $\mathbb{L}^n$. Then the 2nd and 3rd-curvature centers are given by

$$C_2(t) = \alpha(t) + a_2 \left( \frac{\varepsilon_1 \varepsilon_2}{H_1 k_2} \right) V_2(t)$$ (4.2)

and

$$C_3(t) = \alpha(t) + a_2 \left( \frac{\varepsilon_1 \varepsilon_2}{H_1 k_2} \right) V_2(t) + a_3 \left( \frac{\varepsilon_1 \varepsilon_2 \varepsilon_3}{H_1^2 k_2^3} \right) (-H_1 k_2 + H_1 k_2^3) V_3(t)$$ (4.3)

respectively, where $a_j = \pm 1$ and

$$m_2 = \left( \frac{\varepsilon_1 \varepsilon_2}{H_1 k_2} \right), \quad m_3 = \left( \frac{\varepsilon_1 \varepsilon_2 \varepsilon_3}{H_1^2 k_2^3} \right) (-H_1 k_2 + H_1 k_2^3).$$ (4.4)

Proposition 2. Let $\alpha$ be a Frenet curve of osculating order 3 in $\mathbb{L}^3$. If the 2nd curvature center of $\alpha$ lies in osculating plane of $\alpha$, i.e. $C_2(t) = \lambda_1 V_1 + \lambda_3 V_3$ for some functions $\lambda_i$ then

$$\lambda'_1 = 1 - \varepsilon_2 a_2,$$

$$\lambda_1 k_1 - \lambda_3 k_3 = \left( \frac{a_2 \varepsilon_1}{k_1} \right)'$$ (4.5)

$$\lambda'_3 = \frac{a_2 \varepsilon_1 \varepsilon_2 \varepsilon_3}{H_1}$$

where $k_1 \neq 0$ and $k_2 \neq 0$.

Proof : Differentiating $C_2(t) = \lambda_1(t)V_1 + \lambda_3(t)V_3$ with respect to parameter $t$ and using (2.2) we get

$$C'_2(t) = \lambda'_1 V_1 + \lambda_1 \varepsilon_2 k_1 V_2 + \lambda'_3 V_3 + \lambda_3 (-\varepsilon_2 k_2 V_2)$$ (4.6)

Similarly, differentiating (4.2) we may obtain

$$C'_2 = \alpha'(t) + \left( \frac{a_2 \varepsilon_1 \varepsilon_2}{k_1} \right) V_2 + \left( \frac{a_2 \varepsilon_1 \varepsilon_2}{k_1} \right) (-\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_2)$$ (4.7)

Hence, comparing (4.6) with (4.7) we get the result. □

As a consequence of Proposition 11 we have the following result.

Corollary 4. Let $\alpha$ be a helix of osculating order 3 in $\mathbb{L}^3$ such that the 2nd curvature center of $\alpha$ lies in osculating plane of $\alpha$ then

$$\lambda_1 = 2t, \quad a_2 \varepsilon_2 = -1$$

$$\lambda_3 = \lambda_1 H_1,$$ (4.8)

$$H_1^2 = \frac{a_2 \varepsilon_1 \varepsilon_2 \varepsilon_3}{2}, \quad \varepsilon_1 \varepsilon_2 = -1$$

where $a_2 \varepsilon_2 = \varepsilon_1 \varepsilon_2 = -1$.

Proposition 3. Let $\alpha$ be a Frenet curve of osculating order 3 in $\mathbb{L}^3$. If the 2nd curvature center of $\alpha$ lies in normal plane of $\alpha$, i.e. $C_2(t) = \lambda_2 V_2 + \lambda_3 V_3$ for some functions $\lambda_i$ then

$$\begin{cases}
-\lambda_2 \varepsilon_1 k_1 = 1 - a_2 \varepsilon_2, \\
\lambda'_2 - \lambda_3 \varepsilon_2 k_2 = a_2 \varepsilon_1 \varepsilon_2 \left( \frac{1}{k_1} \right)', \\
\lambda_2 \varepsilon_3 k_2 + \lambda'_3 = a_2 \varepsilon_1 \varepsilon_2 \varepsilon_3 \left( \frac{1}{H_1} \right)
\end{cases}$$ (4.9)

where $a_2 \varepsilon_2 \neq 1, k_1 \neq 0$ and $k_2 \neq 0$. 
Proof: Differentiating $C_2(t) = \lambda_2(t)V_2 + \lambda_3(t)V_3$ with respect to $t$ and using (2.2) we may get

$$C'_2 = \lambda'_1 V_2 + \lambda_1(-\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_3) + \lambda'_3 V_3 + \lambda_3(-\varepsilon_2 k_2 V_2).$$  \hspace{1cm} (4.10)

Similarly, comparing (4.7) with (4.10) we get the result. □

As a consequence of Proposition 11 we have the following result.

Corollary 5. There is no helix of osculating order 3 in $\mathbb{L}^3$ whose curvature center lies in normal plane itself.

Proof: Let $\alpha$ be a helix of osculating order 3 in $\mathbb{L}^3$. Then by (4.10) we get

$$\lambda_2 = \frac{a_2 \varepsilon_2 - 1}{\varepsilon_1 k_1},$$

$$\lambda_3 = 0.$$

So $C_2(t)$ can not be written of the form $C_2(t) = \lambda_2(t)V_2 + \lambda_3(t)V_3$. □

Proposition 4. There is no Frenet curve $\alpha$ in $\mathbb{L}^3$ of osculating order 3 which its 2nd curvature center lies in the tangent plane of $\alpha$.

Proof: Let the 2nd curvature center of $\alpha$ lies in tangent plane of $\alpha$ then $C_2(t) = \lambda_1 V_1 + \lambda_2 V_2$. So, differentiating $C_2(t)$ with respect to $t$ and using (2.2) we may get

$$C'_2 = \lambda'_1 V_1 + \lambda_1(\varepsilon_2 k_1 V_2) + \lambda'_2 V_2 + \lambda_2(-\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_3).$$  \hspace{1cm} (4.11)

Furthermore, comparing (4.7) with (4.11) we may obtain

$$\begin{align*}
\lambda'_1 - \lambda_2 \varepsilon_1 k_1 &= 1 - a_2 m_2 \varepsilon_1 k_1, \\
\lambda_1 \varepsilon_2 k_1 + \lambda'_2 &= (a_2 m_2)', \\
\lambda_2 &= a_2 m_2.
\end{align*}$$  \hspace{1cm} (4.12)

But, from 2nd and 3rd equations we may get $\lambda_1 = 0$. Substituting $\lambda_1 = 0$ into (4.12) one can show that the resultant system of differential equation is not consistent.

ÖZET: Bu çalışmada, $L^n$ Lorentz uzayındaki Frenet eğrilerinin harmonik eğrilikleri $H_j$, ($1 \leq j \leq r$) ele alınmıştır. Bununla beraber $L^n$ deki Frenet eğrilerinin $r$-inci eğrilik merkezleri $C_r(t)$ lemin bir karakterizasyonu verilmiştir. Eğrilerin helis olması durum ayrıca incelemiştir.

REFERENCES


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