SOME RESULTS ON NEAR-RINGS WITH GENERALIZED DERIVATIONS

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ABSTRACT. Let $N$ be a prime right near-ring with multiplicative center $Z$, $f: R \rightarrow R$ a generalized derivation associated with derivation $d$. The following results are proved: (i) If $f^2(N) = 0$ then $f = 0$. (ii) If $f(N) \subseteq Z$ then $N$ is commutative ring. (iii) $f(xy) = f(x)f(y)$ or $f(xy) = f(y)f(x)$ for all $x, y \in N$ then $d = 0$.

1. INTRODUCTION

Throughout this paper, $N$ stands for a right near-ring with multiplicative center $Z$. An additive map $d : N \rightarrow N$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ or equivalently (cf. [7]) that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in [2]. The notion of generalized derivation of a prime ring was introduced by M. Bresar and B. Hvala in [4] and [6]. Some recent results concerning commutativity in prime near-rings with derivation have been generalized in several ways. Many authors have investigated these theorems for generalized derivation. It is my purpose to extend some comparable results on near-rings with generalized derivation.

According to [2], a near ring $N$ is said to be prime if $xNy = \{0\}$ for $x, y \in N$ implies $x = 0$ or $y = 0$. For $x, y \in N$ the symbol $(x, y)$ will denote the additive-group commutator $x + y - yx$, while the symbol $[x, y]$ will denote the commutator $xy - yx$. Let $S$ be a nonempty subset of $N$ and $d$ be a derivation of $N$. If $d(xy) = d(x)d(y)$ or $d(xy) = d(y)d(x)$ for all $x, y \in S$, then $d$ is said to act as a homomorphism or anti-homomorphism on $S$, respectively.

In [3], Bell and Kappe proved that if $d$ is a derivation of a semi-prime ring $R$ which is either an endomorphism or anti-endomorphism, then $d = 0$. Argaç extended that above conclusion holds for near-rings in [1].

Two results are obtained in this paper: The first result states that if $f$ is a generalized derivation of $N$ such that $f^2 = 0$ then $f = 0$. The second result proves that $f$ is generalized derivation of a prime near-ring $N$ which is either a homomorphism or an anti-homomorphism on $N$, then $d = 0$. As for terminologies used here without mention, we refer to G. Pilz [8].

We shall give a description of generalized derivation associated with $d$ by motivated [5, Definition 1].

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Definition 1.1. Let \( N \) be a near-ring, \( d \) a derivation of \( N \). An additive mapping \( f : N \rightarrow N \) is said to be right generalized derivation associated with \( d \) if
\[
f(xy) = f(x)y + xd(y) \text{ for all } x, y \in R. \tag{1.1}
\]
and \( f \) is said to be left generalized derivation associated with \( d \) if
\[
f(xy) = dx(y) + xf(y) \text{ for all } x, y \in R. \tag{1.2}
\]
\( f \) is said to be a generalized derivation associated with \( d \) if it is both a left and right generalized derivation associated with \( d \).

Lemma 1.2. [1, Lemma 1] Let \( N \) be a near-ring and \( d \) a derivation of \( N \), then \( a(yd(x) + d(y)x) = ayd(x) + ad(y)x \) for all \( a, x, y \in N \).

Lemma 1.3. Let \( N \) be a prime near-ring, \( d \) a nonzero derivation of \( N \) and \( a \in N \). If \( ad(N) = 0 \) \( (d(N)a = 0) \), then \( a = 0 \).

Proof. Suppose that \( ad(N) = 0 \). For arbitrary \( x, y \in N \), we have
\[
0 = ad(xy) = ad(x)y + axd(y).
\]
By the hypothesis,
\[
axd(y) = 0, \text{ for all } x, y \in N.
\]
Since \( N \) is prime near-ring and \( d \neq 0 \), we get \( a = 0 \).
Similarly argument works if \( d(N)a = 0 \). \( \square \)

Lemma 1.4. Let \( N \) be a 2- torsion free prime near-ring and \( d \) a derivation of \( N \). If \( d^2 = 0 \), then \( d = 0 \).

Proof. For arbitrary \( x, y \in N \), we have
\[
0 = d^2(xy) = d(d(xy)) = d(xd(y) + d(x)y)
\]
\[
= xd^2(y) + 2d(x)d(y) + d^2(x)y.
\]
By the hypothesis,
\[
2d(x)d(y) = 0, \text{ for all } x, y \in N.
\]
Since \( N \) is 2- torsion free near-ring, we get
\[
d(x)d(N) = 0, \text{ for all } x \in N.
\]
Using Lemma 2, we get \( d = 0 \). \( \square \)

Lemma 1.5. Let \( N \) be a prime near-ring and \( d \) a nonzero derivation of \( N \). If \( d(N) \subset Z \), then \((N, +)\) is Abelian. Moreover, if \( N \) is 2-torsion free, then \( N \) is commutative ring.

Proof. Suppose that \( a \in N \) such that \( d(a) \neq 0 \). So, \( d(a) \in Z \setminus \{0\} \) and \( d(a) + d(a) \in Z \setminus \{0\} \). For all \( x, y \in N \), we have
\[
(d(a) + d(a))(x + y) = (x + y)(d(a) + d(a))
\]
that is,
\[
d(a)x + d(a)x + d(a)y + d(a)y = xd(a) + yd(a) + xd(a) + yd(a).
\]
Since \( d(a) \in Z \), we get
\[
xd(a) + yd(a) = yd(a) + xd(a)
\]
and so,
\[
(x, y)d(a) = 0, \text{ for all } x, y \in N.
\]
Since \( d(a) \in Z \setminus \{0\} \) and \( N \) is a prime near-ring, we get \((x, y) = 0\), for all \( x, y \in N \). Thus \((N, +)\) is Abelian.

Now, using the hypothesis, for any \( a, b, c \in N \),
\[
\text{cd}(ab) = d(ab)c.
\]

By Lemma 1, one can obtains
\[
cad(b) + cd(a)b = ad(b)c + d(a)bc.
\]

Using \( d(N) \subset Z \) and \((N, +)\) is Abelian, we obtain that
\[
cad(b) + cba = acd(b) + bcd(a)
\]
which yields
\[
([c, a])d(b) = [b, c]d(a), \quad \text{for all } a, b, c \in N.
\]

Suppose now that \( N \) is not commutative. Choosing \( b, c \in N \) such that \([b, c] \neq 0\)
and replacing \( a \) by \( d(a) \in Z \), we get
\[
[b, c]d^2(a) = 0, \quad \text{for all } a, b, c \in N.
\]
Since \( d^2(a) \in Z \), we conclude that \( d^2(a) = 0 \), for all \( a \in N \), and so \( d = 0 \) by Lemma 3. But it contradicts \( d \neq 0 \). This completes the proof.

**Lemma 1.6.** (i) Let \( f \) be right generalized derivation of \( N \) associated with \( d \). Then \( f(xy) = xd(y) + f(x)y \) for all \( x, y \in N \).

(ii) Let \( f \) be left generalized derivation of \( N \) associated with \( d \). Then \( f(xy) = xf(y) + d(x)y \) for all \( x, y \in N \).

**Proof.** (i) For any \( x, y \in N \), we get
\[
f((x + x)y) = f(x + x)y + (x + x)d(y)
= f(x)y + f(x)y + xd(y) + xd(y)
\]
and
\[
f(xy + xy) = f(x)y + xd(y) + f(x)y + xd(y).
\]
Comparing these equations, one can obtain
\[
f(x)y + xd(y) = xd(y) + f(x)y, \quad \text{for all } x, y \in N.
\]
That is \( f(xy) = xd(y) + f(x)y \).

(ii) Similarly.

**Lemma 1.7.** Let \( f \) be generalized derivation of \( N \) associated with \( d \). Then \( a(xy) + f(x)y = a(xd(y) + a f(x)y \) for all \( a, x, y \in N \).

**Proof.** The proof can be given using a similar approach as in the proof of [2, Lemma 1]. For any \( a, x, y \in N \), we get
\[
f(a(xy)) = af(xy) + d(a)xy = a(xd(y) + f(x)y) + d(a)xy.
\]
On the other hand,
\[
f((ax)y) = axd(y) + f(ax)y
= axd(y) + (a f(x) + d(a)x)y = axd(y) + af(x)y + d(a)xy.
\]
For two expressions of \( f(axy) \), we obtain that
\[
a(xd(y) + af(x)y) = axd(y) + af(x)y, \quad \text{for all } a, x, y \in N.
\]

**Lemma 1.8.** Let \( N \) be a prime near-ring, \( f \) a nonzero generalized derivation of \( N \) associated with nonzero derivation \( d \) and \( a \in N \).
(i) If $af(N) = 0$, then $a = 0$.
(ii) If $f(N)a = 0$, then $a = 0$.

Proof. (i) For all $x, y \in N$, we get
$$0 = af(xy) = axd(y) + af(x)y$$
and so,
$$aNd(N) = 0.$$ 
Since $N$ is prime near-ring and $d \neq 0$, we obtain $a = 0$.

(ii) A similar argument works if $f(N)a = 0$. \qed 

**Theorem 1.9.** Let $f$ be a generalized derivation of $N$ associated with nonzero derivation $d$. If $N$ is a 2-torsion free prime near-ring and $f^2 = 0$, then $f = 0$.

Proof. For arbitrary $x, y \in N$, we have
$$0 = f^2(xy) = f(f(xy)) = f(f(x)y + xd(y))$$
$$= f^2(x)y + 2f(x)d(y) + xd^2(y).$$
By the hypothesis,
$$2f(x)d(y) + xd^2(y) = 0 \text{ for all } x, y \in N. \quad (1.3)$$
Writing $f(x)$ by $x$ in (1.3), we get
$$f(x)d^2(y) = 0 \text{ for all } x, y \in N.$$
By Lemma 7 (ii), we obtain that $d^2(N) = 0$ or $f = 0$. If $d^2(N) = 0$ then $d = 0$ from Lemma 3, a contradiction. So, we find $f = 0$. \qed 

**Theorem 1.10.** Let $N$ be a prime near-ring with a nonzero generalized derivation $f$ associated with nonzero derivation $d$. If $f(N) \subset Z$, then $(N, +)$ is Abelian. Moreover, if $N$ is 2-torsion free, then $N$ is a commutative ring.

Proof. The same argument used in the proof of Lemma 4 shows that both $f(a) \in Z \setminus \{0\}$ and $f(a) + f(a) \in Z \setminus \{0\}$, then we have.
$$f(a)(x, y) = 0 \text{ for all } x, y \in N.$$ 
Since $f(a) \in Z \setminus \{0\}$ and $N$ is a prime near-ring, it follows that $(x, y) = 0$, for all $x, y \in N$. Thus $(N, +)$ is abelian.

Using the hypothesis, for any $x, y, z \in N$,
$$zf(xy) = f(xy)z.$$ 
By Lemma 6, we have
$$z(xd(y) + f(x)y) = (f(x)y + xd(y))z$$
$$zx^2d(y) + zf(x)y = f(x)yz + xd(y)z.$$ 
Using $f(N) \subset Z$ and $(N, +)$ is Abelian, we obtain that
$$zx^2d(y) = xzd(y)z = f(x)yz - zf(x)y$$
and so,
$$zx^2d(y) = xzd(y)z = f(x)[y, z], \text{ for all } x, y, z \in N. \quad (1.4)$$
Substituting $f(y)$ for $y$ in (1.4) and using $f(N) \subset Z$, we get
$$[z, x]d(f(y)) = 0, \text{ for all } x, y, z \in N.$$
Since \( f(y) \in Z \) and so \( d(f(y)) \in Z \), we have
\[
d(f(y)) = 0, \text{ for all } y \in N \text{ or } N \text{ is commutative ring.}
\]

Let assume that \( d(f(y)) = 0 \), for all \( y \in N \). Then
\[
0 = d(f(xy)) = d(d(x)y + xf(y)) = d^2(x)y + d(x)d(y) + d(x)f(y) = 0, \text{ for all } x, y \in N.
\]
Replacing \( y \) by \( yz \) in this equation and using this, we obtain that
\[
0 = d^2(x)yz + d(x)d(yz) + d(x)f(yz)
= d^2(x)yz + d(x)d(y)z + d(x)yf(z) + d(x)f(y)z + d(x)yd(z)
= \{d^2(x)y + d(x)d(y) + d(x)f(y)\}z + 2d(x)yd(z) = 2d(x)yd(z).
\]
Since \( N \) is a 2-torsion free near-ring, we get
\[
d(N)Nd(N) = 0.
\]
Thus, we obtain that \( d = 0 \). It contradicts \( d \neq 0 \). So we must have \( N \) is commutative ring.

**Theorem 1.11.** Let \( N \) be a prime near-ring and \( f \) be a generalized derivation of \( N \) associated with \( d \). If \( f \) acts as a homomorphism on \( N \), then \( d = 0 \).

**Proof.** Let \( f \) acts as a homomorphism on \( N \). Then
\[
f(xy) = f(x)f(y) = xd(y) + f(x)yx, \text{ for all } x, y \in N. \tag{1.5}
\]
Taking \( yx \) by \( y \) in (1.5), we get
\[
xd(yx) + f(x)y = f(x)f(yx) = f(x)(yd(x) + fyx) = f(x)yd(x) + f(x)f(y)x
= f(x)yd(x) + f(xy)x = f(xy)d(x) + xd(y)x + f(xy)x
\]
and so,
\[
xd(yx) = f(x)yd(x) + xd(y)x, \text{ for all } x, y \in N.
\]
Using Lemma 1, we obtain that
\[
xyd(x) = f(x)yd(x), \text{ for all } x, y \in N. \tag{1.6}
\]
Replacing \( f(y) \) by \( y \) in (1.6), then
\[
xf(y)d(x) = f(x)f(y)d(x) = f(xy)d(x) = d(xy)yd(x) + xf(y)d(x)
\]
and so,
\[
d(x)Nd(x) = 0, \text{ for all } x \in N.
\]
Since \( N \) is a prime near-ring, we have \( d = 0 \).

**Theorem 1.12.** Let \( N \) be a prime near-ring and \( f \) be a generalized derivation of \( N \) associated with \( d \). If \( f \) acts as a anti-homomorphism on \( N \), then \( d = 0 \).

**Proof.** By the hypothesis, we get
\[
f(y)f(x) = xd(y) + f(x)y, \text{ for all } x, y \in N. \tag{1.7}
\]
Replacing \( yx \) by \( xy \) in (1.7), then
\[
xyd(y) + f(xy)y = f(y)f(xy) = f(y)(xd(y) + f(x)y) = f(y)xd(y) + f(y)f(x)y
= f(y)xd(y) + f(xy)y
\]
and so
\[
xyd(y) = f(y)xd(y), \text{ for all } x, y \in N. \tag{1.8}
\]
If we take \( rx \) instead of \( x \) in (1.8), we have
\[
f(y)rxd(y) = rxyd(y) = rf(y)xd(y)
\]
and so
\[ [r, f(y)] x d(y) = 0, \] for all \( x, y, r \in N \).

Since \( N \) is a prime near-ring, we arrive at \( f(y) \in Z \) or \( d(y) = 0 \), for all \( y \in N \).

Let’s define \( A = \{ x \in N \mid d(x) = 0 \} \) and \( B = \{ x \in N \mid f(x) \in Z \} \). Clearly each of \( A \) and \( B \) is additive subgroup of \( N \) such that \( N = A \cup B \). But, a group can not be the set-theoretic union of two proper subgroups. Hence \( N = A \) or \( N = B \). In the latter case, \( f(N) \subset Z \), which forces \( f \) acts as homomorphism on \( N \), and so \( d = 0 \) by Theorem 3. If \( N = A \) then \( d = 0 \). The proof is completed. \( \square \)

ÖZET: \( N \) merkezi \( Z \) olan bir sağ asal near-halka, \( f : N \rightarrow N \) tanımlı \( d \) ile ilgili bir genelleştirilmiş türev olsun. Bu durumda: (i) Eğer \( f^2(N) = 0 \) ise \( f = 0 \) dir. (ii) Eğer \( f(N) \subset Z \) ise \( N \) değişmeli bir halkadır. (iii) Eğer her \( x, y \in N \) için \( f(xy) = f(x)f(y) \) veya \( f(xy) = f(y)f(x) \) ise \( d = 0 \) dir.

REFERENCES


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