ON GROUPS WHOSE EVERY PROPER SUBGROUP IS A $B_n$-GROUP

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ABSTRACT
Let $G$ be a group in which every proper subgroup is a $B_n$-group. Also, $[G, X]$ has finite exponent, for all $x \in G$. $B_n$-group denotes the class of all group in which no subnormal subgroups has defect exceeding $n$, where $n$ is a natural number. It is proved that the groups are soluble and Fitting groups.

KEYWORDS: Finite exponent, locally nilpotent groups, $B_n$-groups.

1. INTRODUCTION

Let $n$ be a natural number. The class of all groups in which every subnormal subgroup has defect at most $n$ is denoted by $B_n$. The groups in $B_1$ is studied in [11], [5], [12], and $B_2, B_3, B_4$ studied in [8], [3]. Moreover the general case $B_n$ studied in [7],[4], [6] and [10].

Let $G$ be a group and let $X$ be a property of groups. If every proper subgroup of $G$ satisfies $X$ but itself does not satisfy it, then $G$ is called a minimal non-$X$-group. Minimal non-$B_1$-groups are considered in [13] and given a classification of such groups. See also [2] for some other results related to the groups in which every subgroup is a $B_1$-group.

If we consider a locally nilpotent with every proper subgroup $B_n$-group $G$, where $n$ is fixed, then we can see that $G$ is nilpotent. Therefore, we consider locally nilpotent groups with every proper subgroup $H$ of $G$ a $B_n$-group for some natural number depending on $H$. 
The following definitions are needed in the sequel. A group $G$ is radicable if every elements of $G$ has an $m$th root for every positive integer $m$. Let $l$ denote the class of periodic radicable abelian groups.

A group $G$ is a $\varphi' \cdot l$-group if and only if there is a transfinite ascending series $\{ G_\alpha \}_{\alpha \leq \beta}$ in $G$ with $G = G_\alpha$ and each $G_{\alpha + 1} / G_\alpha$ is an $l$-group.

2. MAIN RESULTS

**Theorem 2.1.** Let $G$ be a locally nilpotent hyperabelian $p$-group for some prime $p$ such that every proper subgroup $H$ of $G$ is a $B_n$-group for some natural number $n$ depending on $H$. If $[G, x]$ has finite exponent for all $x \in G$ then,

(i) $G$ is soluble,

(ii) $G$ is a Fitting group and every subgroup of $G$ is subnormal.

**Proof.** (i) Assume that $G$ is not soluble. Let $H$ be a proper subgroup of $G$ then $H$ is $\varphi' \cdot l$-by-nilpotent for all proper subgroups $H$ of $G$ by Corollary 6.4 [7]. Thus $H$ has a normal $\varphi'$ subgroup $N$ such that $H/N$ is nilpotent. Assume that $N$ is not nilpotent. Since $N$ is hyperabelian $p$-group $N$ has a normal abelian series, $1 = N_0 < N_1 < N_2 < \ldots < N_\beta = N$.

Let $\lambda$ be the least ordinal such that $N_\lambda$ is not nilpotent. If $\lambda$ is a limit ordinal then $N_\lambda = \bigcup_{\mu < \lambda} N_\mu$. Thus $N_\lambda$ is nilpotent for all $\mu < \lambda$. For every $x \in N_\lambda$ there exists $\mu < \lambda$ such that $x \in N_\mu$. Thus $N_\lambda$ is a Fitting group and hence $N_\lambda$ is nilpotent by Lemma 6.1 [7], but this is a contradiction. Thus $\lambda - 1$ exists and $N_{\lambda - 1}$ is nilpotent. Then $N_\lambda$ is a soluble $p$-group. Since $<x>^{N_\lambda}$ is soluble and it has finite exponent by hypothesis. Thus $<x>^{N_\lambda}$ is Baer group by Theorem 7.17 [14]. Now $<x>^{N_\lambda}$ is nilpotent by Lemma 6.1 [7] and hence $H$ is soluble. In addition $<x>^H$ has finite exponent. Therefore $<x>^H$ is a Baer group by Theorem 7.17
[14]. This implies that \( H \) is nilpotent and that \( G \) is a Fitting group by Theorem 3.3 (ii) [16] and by Theorem 1.1 [1] \( G \) is soluble, a contradiction.

(ii) Assume that \( G \) is not a Fitting group. Then \( G \) cannot be nilpotent and hence every proper subgroup of \( G \) is nilpotent and thus \( G \) is soluble by (i). Since \( <x>^G \) has finite exponent for all \( x \in G \), \( <x>^G \) is a Baer group by Theorem 7.17 [14] and hence \( <x>^G \) is nilpotent by Lemma 6.1 [7]. Therefore \( G \) is a Fitting group. Assume that \( G \) has a maximal subgroup \( M \). Then \( M \) is a normal subgroup of \( G \), since \( G \) is locally nilpotent. Now there exists a finitely generated subgroup \( F \) of \( G \) such that \( G=FM \). By Lemma 1 [9] \( G \) is nilpotent. If \( G \) has no maximal subgroup then every subgroup of \( G \) is subnormal by Theorem 3.1. (ii) [16].

Corollary 2.2. Let \( G \) be a periodic locally nilpotent hyperabelian group and let every proper subgroups \( H \) of \( G \) be a \( B_n \)-group for a natural number \( n \) depending on \( H \). If \( [G,x] \) has finite exponent for all \( x \in G \) then,

(i) \( G \) is soluble,

(ii) \( G \) is a Fitting group and every subgroup of \( G \) is subnormal.

Proof. (i) Clearly \( G \) is the direct product of primary components by 12.1.1 [15]. If \( G \) is a \( p \)-group, then \( G \) is soluble by Theorem 2.1. If \( G \) is not a \( p \)-group then every primary components of \( G \) is soluble. Every primary components of \( G \) is nilpotent by the proof of Theorem 2.1. Let \( H \) be a proper subgroup of \( G \). \( <x>^H \) has finite exponent, for all \( x \in G \) by hypothesis. This implies that \( <x>^H \) has finitely primary components. Since every primary components of \( G \) is nilpotent, primary components of \( <x>^H \) is nilpotent. This implies that \( <x>^H \) is nilpotent by 5.2.8[15]. Thus \( H \) is a Baer group. \( H \) is nilpotent by Lemma 6.1.[7]. Thus, every proper subgroup of \( G \) is nilpotent. Therefore, \( G \) is a Fitting \( p \)-group for some prime \( p \) by Theorem 3.3 (i),(ii) [16]. \( G \) is soluble by Theorem 2.1.

(ii) \( G \) is soluble by (i). Also, \( G \) is the direct product of primary components by 12.1.1 [15]. If \( G \) is a \( p \)-group, then \( G \) is a Fitting group by Theorem 2.1. If \( G \) is not a \( p \)-group then every primary components of \( G \) is a Fitting group by Theorem 2.1. Thus every primary components of \( G \) is nilpotent by
Lemma 6.1.[7]. $<x>^G$ has finite exponent, for all $x \in G$ by hypothesis. This implies that $<x>^G$ is has finitely primary components. This primary components is nilpotent. Thus $G$ is a Fitting group. If $G$ has a maximal, then $M$ is a normal subgroup of $G$, since $G$ is locally nilpotent. Now there exists a finitely generated subgroup $F$ of $G$ such that $G = FM$. By Lemma 1 [9] $G$ is nilpotent. If $G$ has no maximal subgroup then every subgroup of $G$ is subnormal by Theorem 3.1. (ii) [16].

**Corollary 2.3.** Let $G$ be a periodic locally nilpotent hyperabelian group and let every proper subgroups $H$ of $G$ be a $B_n$-group for a natural number $n$ depending on $H$. If $G'$ has finite exponent then,

(i) $G$ is soluble,

(ii) $G$ is a Fitting group and every subgroup of $G$ is subnormal.

**Corollary 2.4.** Let $G$ be a locally nilpotent group and let every proper subgroup $H$ of $G$ be a $B_n$-group for a natural number $n$ depending on $H$. If every proper subgroup $H$ of $G$ is soluble and has finite exponent then,

(i) $G$ is soluble,

(ii) $G$ is a Fitting group and every subgroup of $G$ is subnormal.

**ÖZET**


**REFERENCES**


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