A NOTE ON SEIBERG-WITTEN MONOPOLE EQUATIONS ON $\mathbb{R}^8$

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ABSTRACT

Salamon's generalizations of the Seiberg-Witten equations are meaningful on any even-dimensional manifolds. In this work we show that there are no nontrivial solutions of these equations for any spin$^c$-structures on $\mathbb{R}^8$.

1. INTRODUCTION

The Seiberg-Witten monopole equations are stated for 4-dimensional manifolds and these equations have great importance for the topology of smooth four-manifolds (see [7], [5]). There are also some analogous to these equations in 8-dimension (see [2], [7], [3]). In [1] it is shown that the one given by Salamon [7] have no nontrivial solutions for the standard spin$^c$-structures on $\mathbb{R}^8$. In this work we show that Salamon's generalization of the Seiberg-Witten equations have no nontrivial solutions for any spin$^c$-structures on $\mathbb{R}^8$.

2. PRELIMINARIES

In this section we give some basic definitions and facts about Seiberg-Witten monopole equations. For more details one can look in [7].

Definition 2.1. A spin$^c$-structure on a $2n$-dimensional oriented real Hilbert space $V$ is a pair $(\mathcal{W}, \Gamma)$ where $\mathcal{W}$ is a $2^n$-dimensional complex Hermitian vector space and $\Gamma : V \to \text{End}(\mathcal{W})$ is a linear map which satisfies

$$\Gamma(\nu)^* + \Gamma(\nu) = 0, \quad \Gamma(\nu)^* \Gamma(\nu) = \|\nu\|^2$$

for every $\nu \in V$.

It is pointed out in [7] that such a map can be extended to an algebra isomorphism $\text{Cl}(V) \to \text{End}(\mathcal{W})$ which satisfies $\Gamma(\bar{x}) = \Gamma(x)^*$, where
\( \text{Cl}(V) \cong \text{Cl}(V) \otimes \mathbb{C} \) is complex Clifford algebra over \( V \), \( \overline{x} \) is conjugate of \( x \) in \( \text{Cl}(V) \) and \( \Gamma(x)\star \) denotes hermitian-conjugate of \( \Gamma(x) \).

Let \( (W_1, \Gamma_1) \) and \( (W_2, \Gamma_2) \) be two spin\(^c\)-structures on \( V \). If there exists a unitary isomorphism \( U : W_1 \to W_2 \) such that

\[
U \Gamma_1(v) U^* = \Gamma_2(v)
\]

for all \( v \in V \), then the spin\(^c\)-structures \( (W_1, \Gamma_1) \) and \( (W_2, \Gamma_2) \) are said to be equivalent. It is known that such a unitary isomorphism always exists as a result of the following proposition (see [7]).

**Proposition 2.2.** Let \( (W_1, \Gamma_1) \) and \( (W_2, \Gamma_2) \) be two spin\(^c\)-structures on \( V \). Then there exists a unitary isomorphism \( U : W_1 \to W_2 \) such that

\[
U \Gamma_1(v) U^* = \Gamma_2(v)
\]

for all \( v \in V \).

Let \( (W, \Gamma) \) be a spin\(^c\)-structure on \( V \). There is a natural splitting of \( W \).

Fix an orientation of \( V \) and denote by

\[
\epsilon = e_{2n} \cdots e_1 \in \text{Cl}(V)
\]

the unique element of \( \text{Cl}(V) \) which has degree \( 2n \) and is generated by a positively oriented orthonormal basis \( e_1, \cdots, e_{2n} \). Then \( \epsilon^2 = (-1)^n \) and hence

\[
W = W^+ \oplus W^-
\]

where the \( W^\pm \) are the eigen spaces of \( \Gamma(\epsilon) \)

\[
W^\pm = \{ w \in W : \Gamma(\epsilon)w = \pm i^n w \}.
\]

Note that \( \Gamma(v)W^+ \subset W^- \) and \( \Gamma(v)W^- \subset W^+ \) for every \( v \in V \). So the restriction of \( \Gamma(v) \) to \( W^+ \) for \( v \in V \) determines a linear map \( \gamma : V \to \text{Hom}(W^-, W^+) \) which satisfies

\[
\gamma(v)^* \gamma(v) = |v|^2 1
\]

for every \( v \in V \).

Let \( (W, \Gamma) \) be a spin\(^c\)-structure on \( V \). Such a structure gives an action of the space of 2-forms \( \Lambda^2 V \) on \( W \). This action is defined by the following:
Firstly, identify $\Lambda^2 V$ with the space of second order elements of Clifford algebra $C_2(V)$ via the map

$$\Lambda^2 V \rightarrow C_2(V), \eta = \sum_{i<j} \eta_{ij}e_i \wedge e_j \mapsto \sum_{i<j} \eta_{ij}e_i e_j.$$  

Compose this map with $\Gamma$ to obtain a map $\rho: \Lambda^2 V \rightarrow \text{End}(W)$ given by

$$\rho\left(\sum_{i<j} \eta_{ij}e_i \wedge e_j\right) = \sum_{i<j} \eta_{ij} \Gamma(e_i) \Gamma(e_j)$$

for any orthonormal basis $e_1, \cdots, e_{2n}$ of $V$. This map is independent of the choice of the orthonormal basis $e_1, \cdots, e_{2n}$. The spaces $W^\pm$ are invariant under $\rho(\eta)$ for every 2-form $\eta \in \Lambda^2 V$. So we can define

$$\rho^\pm(\eta) = \rho(\eta)|_{W^\pm}$$

for $\eta \in \Lambda^2 V$. In 4-dimensions $\rho^+(\eta) = \rho^+(\eta^+)$ for every 2-form $\eta \in \Lambda^2 V$, where $\eta^+$ is the self-dual part of $\eta$. The map $\rho$ extends to a map

$$\rho: \Lambda^2 V \otimes \mathbb{C} \rightarrow \text{End}(W)$$

on the space of complex valued 2-forms. If $\eta$ is a real valued 2-form, then $\rho(\eta)$ is skew-Hermitian and if $\eta$ is imaginary valued then $\rho(\eta)$ is Hermitian.

Globalizing above $\Gamma$ to 2n-dimensional oriented manifold $X$ defines a spin$^c$ structure $\Gamma: TX \rightarrow \text{End}(W)$, $W$ being a 2$^n$-dimensional complex Hermitian vector bundle on $X$. Such a structure exists if $w_2(X)$ has an integral lift (see [4]). $\Gamma$ extends to an isomorphism between the complex Clifford algebra bundle $\mathcal{C}l(TX)$ and $\text{End}(W)$. There is a natural splitting $W = W^+ \oplus W^-$ into the $\pm i^n$ eigenspaces of $\Gamma(e_{2n} e_{2n-1} \cdots e_1)$ where $e_1, e_2, \cdots e_{2n}$ is any positively oriented local orthonormal frame of $TX$.

A Hermitian connection $\nabla$ on $W$ is called a spin$^c$ connection (compatible with the Levi-Civita connection) if

$$\nabla_\psi (\Gamma(w) \Psi) = \Gamma(w) \nabla_\psi \Psi + \Gamma(\nabla_\psi w) \Psi$$

where $\Psi$ is a spinor (section of $W$), $\psi$ and $w$ are vector fields on $X$ and $\nabla_\psi w$ is the Levi-Civita connection on $X$. $\nabla$ preserves the subbundles $W^\pm$. 
There is a principal Spin^c(2n)-bundle \( P \) on \( X \) such that the bundle \( W \) of spinors, the tangent bundle \( TX \), and the line bundle \( L_\Gamma \) can be recovered as the associated bundles

\[
W = P \times_{\text{Spin}^c(2n)} \mathbb{C}^{2n}, \quad TX = P \times_{Ad} \mathbb{R}^{2n}
\]

where \( Ad \) is the adjoint action of

\[
\text{Spin}^c(2n) = \{ e^{i\theta} x : \theta \in \mathbb{R}, x \in \text{Spin}(2n) \} \subset \mathbb{C}l_{2n}
\]

on \( R^{2n} \). Then one can obtain a complex line bundle \( L_\Gamma = P \times_\delta \mathbb{C} \) where

\[
\delta : \text{Spin}^c(2n) \rightarrow S^1 \text{ by } \delta(e^{i\theta} x) = e^{2i\theta}.
\]

There is a one-to-one correspondence between spin^c connections on \( W \) and \( \text{spin}^c(2n) = \text{Lie}(\text{Spin}^c(2n)) = \text{spin}(2n) \oplus i\mathbb{R} \)-valued connection 1-forms \( \tilde{A} \in A(P) \subset \Omega^1(P, \text{spin}^c(2n)) \) on \( P \). Hence every spin^c connection \( \tilde{A} \) decomposes as

\[
\tilde{A} = \tilde{A}_0 + \frac{1}{2^n} \text{trace}(\tilde{A})
\]

where \( \tilde{A}_0 \) is the traceless part of \( \tilde{A} \). Let \( A = \frac{1}{2^n} \text{trace}(\tilde{A}) \). This is an imaginary valued 1-form in \( \Omega^1(P, i\mathbb{R}) \) which satisfies

\[
A_{pq}(vg) = A_p(v), \quad A_p(p, \xi) = \frac{1}{2^n} \text{trace}(\xi)
\]

(1)

for \( v \in T_p P \), \( g \in \text{Spin}^c(2n) \), and \( \xi \in \text{spin}^c(2n) \). Let

\[
\mathbf{A}(\Gamma) = \{ A \in \Omega^1(P, i\mathbb{R}) : A \text{ satisfies (1)} \}
\]

There is a one-to-one correspondence between these 1-forms and spin^c connections on \( W \). Let \( \nabla_A \) be the spin^c connection corresponding to \( A \). \( \mathbf{A}(\Gamma) \) is an affine space with parallel vector space \( \Omega^1(X, i\mathbb{R}) \). Let \( F_A \in \Omega^2(P, i\mathbb{R}) \) be the curvature of the 1-form \( A \) and \( D_A \) denote the Dirac operator corresponding to \( A \in \mathbf{A}(\Gamma) \),

\[
D_A : C^\infty(X, W^+) \rightarrow C^\infty(X, W^-)
\]

defined by
where $\Psi \in C^\infty(X, W^+)$ and $e_1, e_2, \ldots, e_{2n}$ is any local orthonormal frame.

The Seiberg-Witten equations can now be expressed as follows:

Let $\Gamma : TX \to \text{End}(W)$ be a fixed spin$^c$ structure on $X$ and consider the pair $(A, \Psi) \in \Lambda(\Gamma) \times C^\infty(X, W^+)$. The Seiberg-Witten equations read

$$D_A \Psi = 0, \quad \rho^+(F_A) = \left(\Psi \Psi^*\right)_0$$

where $(\Psi \Psi^*)_0 \in C^\infty(X, \text{End}(W^+))$ is defined by $(\Psi \Psi^*)(x) = \langle \Psi(x), \tau \rangle \Psi^*(x)$ for $\tau \in C^\infty(X, W^+)$ and $(\Psi \Psi^*)_0$ is the traceless part of $(\Psi \Psi^*)$.

3. MONOPOLE EQUATIONS ON $\mathbb{R}^8$ WITH DIFFERENT Spin$^c$-STRUCTURES AND THEIR RELATIONS

One can find the explicit expressions of the Seiberg-Witten monopole equations on $\mathbb{R}^4$ in [6] and [7].

In our case $X = \mathbb{R}^8$, $W_1 = W_2 = C^{16}$ and $L_T = \mathbb{R}^8 \times C$, $(W_1, \Gamma_1)$ and $(W_2, \Gamma_2)$ spin$^c$-structures on $\mathbb{R}^8$ and we consider the unitary map $U$ from $W_1$ to $W_2$ that satisfies

$$U \circ \Gamma_1(v) \circ U^* = \Gamma_2(v)$$

for all $v \in \mathbb{R}^8$.

In [1] they consider standard spin$^c$-structure which is obtained from the well-known isomorphism of the complex Clifford algebra $Cl_{2n}$ with $\text{End}(\Lambda^* C^n)$ and they express following theorem:

**Theorem 3.1.** There are no nontrivial solutions of the Seiberg-Witten equations on $\mathbb{R}^8$ with constant standard spin$^c$-structure, i.e. $\rho^+(F_A) = \left(\Psi \Psi^*\right)_0$ (alone) implies $F_A = 0$ and $\Psi = 0$. 
Our goal is to state a similar theorem for any spin$^c$-structure on $\mathbb{R}^8$. To do this we need some lemmas.

**Lemma 3.2.** If a unitary isomorphism $U$ from $W_1$ to $W_2$ satisfies (2), then $U$ maps $W_1^\pm$ onto $W_2^\pm$.

**Proof.** Let $\Psi \in C^\infty(\mathbb{R}^8, W_1^+)$. Then $\Gamma_1(\varepsilon)\Psi = \Psi$ where

$$\varepsilon = e_2 \cdots e_n e_1.$$ $\Psi = \Gamma_1(e_2 \cdots e_1)\Psi = \Gamma_1(e_2)\cdots \Gamma_1(e_1)\Psi = U^*\Gamma_2(e_2)\cdots U^*\Gamma_2(e_1)U\Psi = U^*\Gamma_2(e_2 \cdots e_1)U\Psi.$$

From the last equality $\Gamma_2(e_2 \cdots e_1)U\Psi = U\Psi$ that is, $U\Psi \in C^\infty(\mathbb{R}^8, W_2^+)$. Thus $U$ maps $W_1^+$ onto $W_2^+$. It can be shown in a similar way that $U$ maps $W_1^-$ onto $W_2^-$. 

**Lemma 3.3.** The maps $\rho_1 : \Lambda^2(T^*\mathbb{R}^8) \otimes \mathbb{C} \to End(W_1)$ and $\rho_2 : \Lambda^2(T^*\mathbb{R}^8) \otimes \mathbb{C} \to End(W_2)$ satisfy $\rho_1(\eta) = U^*\rho_2(\eta)U^*$ for any 2-form $\eta = \sum \eta_{ij}e_i \wedge e_j$ in $\Lambda^2(T^*\mathbb{R}^8) \otimes \mathbb{C}$.

**Proof.**

$$\rho_2(\eta) = \sum_{i<j} \eta_{ij} \Gamma_2(e_i)\Gamma_2(e_j) = \sum_{i<j} \eta_{ij}U\Gamma_2(e_i)U^* \Gamma_2(e_j)U^* (\text{Since } UU^* = I) = \sum_{i<j} U\eta_{ij} \Gamma_2(e_i)\Gamma_2(e_j)U^* = U\left(\sum_{i<j} \eta_{ij} \Gamma_2(e_i)\Gamma_2(e_j)\right)U^* = U(\rho_1(\eta))U^*.$$ 

Note that $\rho_2^*(\eta) = U(\rho_1^*(\eta))U^*$.
Lemma 3.4. If $\Psi \in C^\infty(\mathbb{R}^8, W_1^*)$, then the equality
\[
((U\Psi)(U\Psi)^*)_0 = U(\Psi\Psi^*)_0 U^*
\]
holds for any unitary isomorphism $U : C^{16} \to C^{16}$.

Proof.
\[
(U(\Psi\Psi^*)_0 U^*)_\tau = (U(\Psi\Psi^*)_0 U^*) \tau
= U(\Psi, U^* \tau) \Psi - \text{trace}(\Psi\Psi^*) U^* \tau
= \langle \Psi, U^* \tau \rangle U \Psi - \text{trace}(\Psi\Psi^*) \tau
= \langle \Psi, U^* \tau \rangle U \Psi - \text{trace}(U \Psi (U \Psi)^*)
= (U \Psi (U \Psi)^*)_0 \tau
\]
for all $\tau \in C^\infty(\mathbb{R}^8, W_1^*)$. Note that,
\[
\text{trace}(\Psi\Psi^*) = \|\Psi\|^2 = \|U\Psi\|^2 = \text{trace}(U \Psi (U \Psi)^*),
\]
since $U$ is unitary.

Lemma 3.5. Let $(\Gamma_1, W_1)$ and $(\Gamma_2, W_2)$ be two spin$^c$-structures on $\mathbb{R}^8$ and $U : W_1 \to W_2$ be a unitary isomorphism such that $U \circ \Gamma_1(\nu) \circ U^* = \Gamma_2(\nu)$ for all $\nu \in \mathbb{R}^8$. If the pair $(A, \Psi)$ is a solution of the monopole equations with respect to $\Gamma_1$, then the pair $(A, U \Psi)$ is a solution of the monopole equations with respect to $\Gamma_2$.

Proof. Let $(A, \Psi)$ be a solution of the equations
\[
D_A \Psi = \sum_{i=1}^8 \Gamma_1(e_i) \nabla_i(\Psi) = 0,
\]
\[
\rho_1^+(F_A) = \sum_{i<j} F_{ij} \Gamma_1(e_i) \Gamma_1(e_j) = (\Psi\Psi^*)_0
\]
Then
\[ D_A(\Psi') = \sum_{i=1}^{8} \Gamma_2(e_i)\nabla_i(\Psi') \]
\[ = \sum_{i=1}^{8} U\Gamma_1(e_i)U^*\nabla_i(\Psi') \]
\[ = \sum_{i=1}^{8} U\Gamma_1(e_i)U^*U\nabla_i(\Psi') \] (since \( \nabla_i(\Psi') = U\nabla_i(\Psi') \))
\[ = U\sum_{i=1}^{8} \Gamma_1(e_i)\nabla_i(\Psi') = U(D_A\Psi) = 0. \]

The equality \( \nabla_i(\Psi') = U\nabla_i(\Psi') \) holds for all \( \Psi' \in C^\infty(\mathbb{R}^8, W^+_1) \),

\[ U\Psi = \left( \sum_{i=1}^{16} u_{1i}\Psi_i, \cdots, \sum_{i=1}^{16} u_{(16)i}\Psi_i \right) \] where \( U = (u_{ij}) \) is the matrix notation of the unitary map \( U \).
\[ \nabla_i (U \Psi) = \nabla_i \begin{bmatrix}
    u_{11} \psi_1 + \cdots + u_{1(16)} \psi_{(16)} \\
    u_{21} \psi_1 + \cdots + u_{2(16)} \psi_{(16)} \\
    \vdots \\
    u_{(16)i} \psi_1 + \cdots + u_{(16)(16)} \psi_{(16)}
\end{bmatrix} \\
= \begin{bmatrix}
    \frac{\partial}{\partial x_i} \left( u_{11} \psi_1 + \cdots + u_{1(16)} \psi_{(16)} \right) + \mathcal{A}_i \left( u_{11} \psi_1 + \cdots + u_{1(16)} \psi_{(16)} \right) \\
    \frac{\partial}{\partial x_i} \left( u_{21} \psi_1 + \cdots + u_{2(16)} \psi_{(16)} \right) + \mathcal{A}_i \left( u_{21} \psi_1 + \cdots + u_{2(16)} \psi_{(16)} \right) \\
    \vdots \\
    \frac{\partial}{\partial x_i} \left( u_{(16)i} \psi_1 + \cdots + u_{(16)(16)} \psi_{(16)} \right) + \mathcal{A}_i \left( u_{(16)i} \psi_1 + \cdots + u_{(16)(16)} \psi_{(16)} \right)
\end{bmatrix} \\
= \begin{bmatrix}
    u_{11} \frac{\partial \psi_1}{\partial x_i} + \cdots + u_{1(16)} \frac{\partial \psi_{(16)}}{\partial x_i} + u_{11} \mathcal{A}_i \psi_1 + \cdots + u_{1(16)} \mathcal{A}_i \psi_{(16)} \\
    u_{21} \frac{\partial \psi_1}{\partial x_i} + \cdots + u_{2(16)} \frac{\partial \psi_{(16)}}{\partial x_i} + u_{21} \mathcal{A}_i \psi_1 + \cdots + u_{2(16)} \mathcal{A}_i \psi_{(16)} \\
    \vdots \\
    u_{(16)i} \frac{\partial \psi_1}{\partial x_i} + \cdots + u_{(16)(16)} \frac{\partial \psi_{(16)}}{\partial x_i} + u_{(16)i} \mathcal{A}_i \psi_1 + \cdots + u_{(16)(16)} \mathcal{A}_i \psi_{(16)}
\end{bmatrix} \\
= \begin{bmatrix}
    u_{11} & u_{12} & \cdots & u_{1(16)} \\
    u_{21} & u_{22} & \cdots & u_{2(16)} \\
    \vdots & \vdots & \ddots & \vdots \\
    u_{(16)i} & u_{(16)2} & \cdots & u_{(16)(16)}
\end{bmatrix} \begin{bmatrix}
    \frac{\partial \psi_1}{\partial x_i} + \mathcal{A}_i \psi_1 \\
    \frac{\partial \psi_2}{\partial x_i} + \mathcal{A}_i \psi_2 \\
    \vdots \\
    \frac{\partial \psi_{(16)}}{\partial x_i} + \mathcal{A}_i \psi_{(16)}
\end{bmatrix}
\]

For the second equation:
\[ \rho_2^+ (F_A) = U \left( \rho_1^+ (\eta) U^* \right) \text{ (from Lemma)} \]
\[ = U \left( \Psi \Psi^* \right)_0 U^* \text{ (since } \Psi \text{ is a solution)} \]
\[ = \left( (U \Psi)(U \Psi)^* \right)_0 \text{ (from Lemma)} \]

To summarise, we can express the following theorem:
Theorem 3.6. Let \( (\Gamma, W) \) be any spin\(^c\)-structure on \( \mathbb{R}^8 \). Then there are no nontrivial solutions of the Seiberg-Witten equations on \( \mathbb{R}^8 \) with arbitrary spin\(^c\)-structure, i.e. \( \rho^+ (F_A) = (\Psi \Psi^*)_0 \) implies \( F_A = 0 \) and \( \Psi = 0 \).

Proof. Let \( (A, \Psi) \) be a solution to the Seiberg-Witten equations on \( \mathbb{R}^8 \) with respect to \( (\Gamma, W) \). Since standard spin\(^c\)-structure is equivalent to the any spin\(^c\)-structure \( (\Gamma, W) \), there exists a unitary isomorphism \( U \) which satisfies the equation (2). Then the pair \( (A, U\Psi) \) is a solution for the Seiberg-Witten equations on \( \mathbb{R}^8 \) with respect to standard spin\(^c\)-structure and from Theorem 3.1., \( A = 0 \) and \( U\Psi = 0 \). Since \( U \) is a isomorphism we get \( \Psi = 0 \).

ÖZET
Salamon’un genelleştirdiği Seiberg-Witten denklemleri herhangi bir çift boyutta anlamılır. Bu çalışmada \( \mathbb{R}^8 \) üzerindeki herhangi bir spin\(^c\) yapısı için Salamon tarafından verilen Seiberg-Witten denklemlerinin nontrivial çözümünün olmadığı gösterilmiştir.

REFERENCES


