BOUNDS FOR PRODUCT OF SINGULAR VALUES USING ROW
(COLUMN) NORM AND DETERMINANT

A. D. GÜNGÖR(1) AND D. TAŞÇI(2)

(1) Department of Mathematics, Art and Science Faculty, Selçuk University, KONYA
(2) Department of Mathematics, Art and Science Faculty, Gazi University, ANKARA

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ABSTRACT

Let $A$ be a $n \times n$ complex matrix with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ and let $1 \leq k < l \leq n$. Bounds for $\sigma_1 \ldots \sigma_k$, $\sigma_k \ldots \sigma_l$, and $\sigma_{n-k+1} \ldots \sigma_n$, involving $k$, $n$, $r_i(c_i)$, $\alpha_i$, and $\det A$, where $r_i(c_i)$ is the Euclidean norm of the $i$-th row (column) of $A$, and $\alpha_i$'s are positive real numbers such that $\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2 = n$, are presented.

1. INTRODUCTION AND PRELIMINARIES

Let $A$ be $n \times n$ an complex matrix. Let $\sigma_i(A)$'s be the singular values of $A$ such that

$$\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_n(A).$$

It is well known that

$$\sigma_1^2(A) + \sigma_2^2(A) + \ldots + \sigma_n^2(A) = \|A\|_F^2$$

and

$$\sigma_1(A)\sigma_2(A)\ldots\sigma_n(A) = |\det A|$$

where $\|A\|_F$ and $\det A$ denote the Frobenius norm of $A$ and the determinant of $A$, respectively.

In Section 2, we have obtained bounds for products of singular values using bounds for eigenvalues in [3] and (1.1) and (1.2) inequalities. In Section 3, we have found bounds for products of singular values using row (column) norm and determinant.

Firstly, we give some preliminaries related to our study.

We define

\[
D = \text{diag} \left\{ \frac{\alpha_1}{r_1(A)}, \frac{\alpha_2}{r_2(A)}, \ldots, \frac{\alpha_n}{r_n(A)} \right\},
\]

where \( r_i(A) \) is the Euclidean norm of the \( i \)-th row of \( A \) and \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are positive real numbers such that

\[
\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2 = n.
\]

Clearly, the Euclidean norm of the coefficient matrix \( B = DA \) of the scaled system is equal to \( \sqrt{n} \) and if \( \alpha_1 = \alpha_2 = \ldots = \alpha_n = 1 \) then each row of \( B \) is a unit vector in the Euclidean norm. Also, we can define \( B = AD \),

\[
D = \text{diag} \left\{ \frac{\alpha_1}{c_1(A)}, \frac{\alpha_2}{c_2(A)}, \ldots, \frac{\alpha_n}{c_n(A)} \right\},
\]

where \( c_i(A) \) is the Euclidean norm of the \( i \)-th column of \( A \). Again, \( \|B\|_F = \sqrt{n} \) and if \( \alpha_1 = \alpha_2 = \ldots = \alpha_n = 1 \) then each column of \( B \) is a unit vector in the Euclidean norm.

**Theorem 1** [2] Let \( A, B \in \mathbb{R}^{m \times n} \) and let \( 1 \leq i_1 < \ldots < i_k \leq n \). Then

\[
\prod_{t=1}^k \sigma_{i_t}(AB) \leq \prod_{t=1}^k \sigma_{i_t}(A) \sigma_{i_t}(B)
\]

and

\[
\prod_{t=1}^k \sigma_t(AB) \geq \prod_{t=1}^k \sigma_{i_t}(A) \sigma_{n-i_t+1}(B).
\]

**Theorem 2** [1] Let \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times m} \) and let \( 1 \leq i_1 < \ldots < i_k \leq n \). Then

\[
\prod_{t=1}^k \sigma_{i_t}(AB) \geq \prod_{t=1}^k \sigma_{i_t}(A) \sigma_{n-i_t+1}(B).
\]
2. BOUNDS FOR PRODUCT OF SINGULAR VALUES USING NORM AND DETERMINANT

We will give bounds that we obtained for singular values as a result of some bounds for eigenvalues which is obtained in [3] where $A$ be a square matrix with singular values

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0.$$ 

**Corollary 3** Let $1 \leq k \leq l \leq n$. Then

$$\left( \frac{k - 1}{\|A\|_F^2} \right)^{(k-1)/2} \left| \det A \right|^{1/n} \leq \sigma_k \ldots \sigma_l \leq \left( \frac{\|A\|_F^2}{l} \right)^{1/2} \left( \frac{1}{\left| \det A \right|} \right)^{(k-1)/n}.$$ 

**Corollary 4** Let $1 \leq k \leq n - 2$. Then

$$\sigma_1 \ldots \sigma_k \leq \frac{1}{(\det A)^2} \left( \frac{1}{n - k} \left( \frac{\|A\|_F^2}{k + 1} \right)^{k+1} \right)^{(n-k-1)/2(n-k-1)}.$$ 

Let $2 \leq k \leq n - 1$. Then

$$\sigma_{n-k+1} \ldots \sigma_n \geq k(\det A)^2 \left( \frac{n - k + 1}{\|A\|_F^2} \right)^{(n-k+1)/2(k-1)}.$$ 

**Corollary 5** Let $1 \leq k \leq l \leq n - 2$. Then

$$\left\{ \left| \det A \right|^{2(1 - \frac{k}{n}) l^2} \left[ (n - k + 1) \left( \frac{k}{\|A\|_F^2} \right)^{n-k+1} \right]^{1/(n-k)} \right\} \leq \sigma_k \ldots \sigma_l.$$
\[
\left( \frac{1}{\det A} \right)^{2[(n-l)k+(l+1)]} \left\{ \frac{1}{n-l} \left( \frac{\|A\|_F^2}{l+1} \right)^{l+1} n^{-l-1} \right\}^{1/2(n-l-1)}
\]

and

\[
(n-k+1)(\det A)^2 \left( \frac{k}{\|A\|_F^2} \right)^{(l-k+1)/2(n-k)} \leq \sigma_k \cdots \sigma_l
\]

\[
\leq \left( \frac{1}{\det A} \right)^{2l[(n-l)k+(l+1)]} \left\{ \frac{1}{n-l} \left( \frac{\|A\|_F^2}{l+1} \right)^{l+1} n^{-l-1} \right\}^{(l-k+1)/2l(n-l-1)}
\]

3. BOUNDS FOR PRODUCT OF SINGULAR VALUES USING ROW (COLUMN) NORM AND DETERMINANT

Since the matrices \(PA, AP\) and \(A\) have the same singular values for any permutation matrix \(P\), we assume for this section, without loss of generality, that the rows and columns of \(A\) are such that

\[
(3.1) \quad r_1(A) \leq r_2(A) \leq \ldots \leq r_n(A)
\]

\[
(3.2) \quad c_1(A) \leq c_2(A) \leq \ldots \leq c_n(A)
\]

and let

\[
(3.3) \quad 0 < \alpha_n \leq \ldots \leq \alpha_2 \leq \alpha_1
\]

for \(\alpha_i\)'s in (1.4).

**Theorem 6** Let \(2 \leq k \leq n-1\). Then

\[
(3.4) \quad \sigma_1 \cdots \sigma_k \geq \prod_{j=n-k+1}^{n} \frac{r_j}{\alpha_j} \left\{ \frac{k}{\prod_{i=1}^{n-k+1} \alpha_i} \left( \frac{\det A}{r_i} \right)^2 \left( \frac{n-k+1}{n} \right)^{n-k+1} \right\}^{1/2(k-1)}
\]

where \(r_i\)'s and \(\alpha_i\)'s be as in (3.1) and (3.3), respectively.
Proof. We write inequality (2.3) for matrix $B=DA$ which is defined in Section 1. Taking $i_1 = n-k+1$ and $i_k = n$ in (1.6) and applying $B = DA$, we obtain (3.4).

Theorem 7 Let $1 < k \leq n - 2$. Then

(3.5) \[ \sigma_1 \cdots \sigma_k \leq \prod_{i=n-k+1}^{n} \frac{\min(r_i, c_i)}{\alpha_i} \left\{ \left( \prod_{i=1}^{n} \frac{\min(r_i, c_i)}{\alpha_i} \right)^2 \left( \frac{1}{(n-k)(k+1)} \right)^{k-1} \right\} \frac{1}{\sqrt{2(n-k-1)}} \]

where $r_i(c_i)$'s and $\alpha_i$'s be as in (3.1) ((3.2)) and (3.3), respectively.

Proof. Write (2.2) for matrix $B=DA$ which is defined in Section 1. Taking $i_1 = n-k+1$ and $i_k = n$ in (1.7) and applying matrix $B=DA$. Then taking $i_1 = 1$ and $i_k = k$ in (1.8) and applying matrix $B=AD$ which is defined in Section 1, we have (3.5).

Theorem 8 Let $1 < k \leq n - 2$. Then

(3.6) \[ \sigma_{n-k+1} \cdots \sigma_n \leq \prod_{i=1}^{k} \frac{\min(r_i, c_i)}{\alpha_i} \left\{ \left( \prod_{i=1}^{n} \frac{\min(r_i, c_i)}{\alpha_i} \right)^2 \left( \frac{1}{(n-k)(k+1)} \right)^{k-1} n-k \right\} \frac{1}{\sqrt{2(n-k-1)}} \]

where $r_i(c_i)$'s and $\alpha_i$'s be as in (3.1) ((3.2)) and (3.3), respectively.

Proof. First, we write (2.2) according to matrix $B$ such that $\|B\|_F = \sqrt{n}$. Taking $i_1 = 1$, $i_k = k$ and $i_1 = n-k+1$, $i_k = n$ in (1.7) and applying matrices $B=DA$ and $B = AD$ which are defined in Section 1, respectively, we get (3.6).
Theorem 9 Let $1 \leq k < l \leq n$. Then

\begin{equation}
\prod_{i=1}^{l-k+1} \frac{c_i}{\alpha_i} \left( \prod_{i=1}^{n} \frac{\alpha_i}{c_i} \right)^2 \left( \frac{k}{n} \right)^{(l-k+1)/2(n-k)} \leq \sigma_k \ldots \sigma_l
\end{equation}

\begin{equation}
\leq \prod_{i=n-l+k}^{n} \frac{c_i}{\alpha_i} \left( \prod_{i=1}^{n} \frac{c_i}{\alpha_i} \right)^2 \left[ \frac{1}{n-l} \left( \frac{n}{l+1} \right)^{l+1} \right]^{n-l} \right)^{(l-k+1)/2(n-l-1)}
\end{equation}

where $c_i$'s and $\alpha_i$'s be as in (3.2) and (3.3), respectively.

Proof. We write inequality (2.1) for matrix $B = AD$ which is defined in Section 1. Taking $i_1 = k$, $i_k = l$ in (1.6) and applying matrix $B = AD$, we obtain (3.7).

Theorem 10 Let $1 \leq k < l \leq n$. Then

\begin{equation}
\sigma_{n-l+1} \ldots \sigma_{n-k+1} \leq \prod_{i=k}^{l} \frac{r_i}{\alpha_i} \left( \prod_{i=1}^{n} \frac{r_i}{\alpha_i} \right)^2 \left[ \frac{1}{n-l+k-1} \left( \frac{n}{l-k+2} \right)^{l-k+2} \right]^{n-l+k-1} \right)^{1/2(n-l+k-2)}
\end{equation}

where $r_i$'s and $\alpha_i$'s be as in (3.1) and (3.3), respectively.

Proof. Firstly, we write inequality (2.3) for matrix $B = DA$ which is defined in Section 1. Then taking $i_1 = k$, $i_k = l$ in (1.7) and applying matrix $B = DA$, we have (3.8).
Theorem 11 Let \( 1 \leq k < l \leq n - 2 \). Then
\[
\sigma_1 \cdots \sigma_{l-k+1} \geq \prod_{i=k}^{l} \frac{r_i}{\alpha_i} \left\{ \left( \prod_{i=1}^{n} \frac{\alpha_i}{r_i} \right)^{2(l-k)/l+2} \left( \frac{n}{k} \right)^{l-k+1} \right\}^{\frac{1}{2(n-k)}}
\]
where \( r_i \)'s and \( \alpha_i \)'s be as in (3.1) and (3.3), respectively.

Proof. Write inequality (2.4) for matrix \( B = DA \) which is defined in Section 1. Taking \( i_1 = k \), \( i_k = l \) in (1.6) and applying matrix \( B = DA \), we get (3.9)

Theorem 12 Let \( 1 \leq k < l \leq n - 2 \). Then
\[
\sigma_{n-l+k} \cdots \sigma_n \leq \prod_{i=k}^{l} \frac{r_i}{\alpha_i} \left\{ \left( \prod_{i=1}^{n} \frac{r_i}{\alpha_i} \right)^{2} \left( \frac{1}{n-l} \left( \frac{n}{l+1} \right)^{l+1} \right)^{l-i} \right\}^{\frac{(l-k+1)}{2l(n-l-1)}}
\]
where \( r_i \)'s and \( \alpha_i \)'s be as in (3.1) and (3.3), respectively.

Proof. To show (3.10), we write (2.5) for matrix \( B = DA \) which is defined in Section 1. Taking \( i_1 = k \), \( i_k = l \) in (1.8) and applying matrix \( B = DA \), we get (3.10).
Theorem 13 Let \( 1 \leq k < l \leq n - 2 \). Then

\[
\sigma_k \ldots \sigma_l \leq \left( \prod_{i=n-l+1}^{n-k+1} \frac{c_i}{\alpha_i} \right) \left( \frac{n}{\det A} \right)^2 \left[ \frac{1}{n-l+k-1} \left( \frac{n}{l-k+2} \right)^{l-k+2} \right]^{\frac{k}{2(n-l+k-2)}}
\]

where \( c_i \)'s and \( \alpha_i \)'s be as in (3.2) and (3.3), respectively.

Proof. To show (3.11), write inequality (2.2) for matrix \( B = AD \) which is defined in Section 1. Taking \( i_1 = k \), \( i_k = l \) in (1.7) and applying matrix \( B = AD \), we get (3.11).

Theorem 14 Let \( 2 \leq k \leq n - 1 \). Then

\[
\sigma_{n-k+1} \ldots \sigma_n \geq \prod_{i=1}^{k} \frac{c_i}{\alpha_i} \left( \prod_{i=1}^{n} \frac{\alpha_i}{c_i} \right) \left( \frac{n-k+1}{n} \right)^{n-k+1} \left( \frac{n}{\det A} \right)^{\frac{k}{2(k-1)}}
\]

where \( c_i \)'s and \( \alpha_i \)'s be as in (3.2) and (3.3), respectively.

Proof. Write inequality (2.3) for matrix \( B = AD \) which is defined in Section 1. Taking \( i_1 = n-k+1 \), \( i_k = n \) in (1.6) and applying matrix \( B = AD \), we have inequality (3.12).
Example 15 Let \( A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 2 \end{bmatrix} \). Singular values of matrix \( A \) are \( \sigma_1 = 6.673 \), \( \sigma_2 = 1.949 \), \( \sigma_3 = 1.181 \) and \( \sigma_4 = 0.521 \). In the following, the best bounds are underlined.

<table>
<thead>
<tr>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \sigma_1 )</td>
<td>( \sigma_1 \sigma_2 )</td>
<td>( \sigma_1 \sigma_2 \sigma_3 )</td>
<td>( \sigma_1 \sigma_2 \sigma_3 \sigma_4 )</td>
</tr>
<tr>
<td>2</td>
<td>( \sigma_2 )</td>
<td>( \sigma_2 \sigma_3 )</td>
<td>( \sigma_2 \sigma_3 \sigma_4 )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( \sigma_3 )</td>
<td>( \sigma_3 \sigma_4 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( \sigma_4 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For product of singular values, we have the following:

1. THE VALUES OF PRODUCT OF SINGULAR VALUES

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.673</td>
<td>13.008</td>
<td>15.362</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>1.949</td>
<td>2.302</td>
<td>1.199</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.181</td>
<td>0.615</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>0.521</td>
<td></td>
</tr>
</tbody>
</table>
2. LOWER BOUNDS FOR PRODUCT OF SINGULAR VALUES:

\[(3.4)\]

\[
\begin{array}{cccc}
  & 1 & 2 & 3 & 4 \\
1 & 0.02 & 0.415 & 2.063 \\
\end{array}
\]

\[(3.7)\]

\[
\begin{array}{cccc}
  & 1 & 2 & 3 & 4 \\
1 & 0.223 & 0.267 & 2.017 \\
2 & 0.032 & 0.038 & \\
3 & 0.002 & \\
\end{array}
\]

\[(3.9)\]

\[
\begin{array}{cccc}
  & 1 & 2 & 3 & 4 \\
1 & 0.198 & 0.415 & 2.063 \\
\end{array}
\]

\[(3.12)\]

\[
\begin{array}{cccc}
  & 1 & 2 & 3 & 4 \\
1 & 2.017 & & \\
2 & 0.038 & & \\
3 & 0.002 & & \\
\end{array}
\]
3. UPPER BOUNDS FOR PRODUCT OF SINGULAR VALUES:

(3.4)

\[
\begin{array}{ccc}
  & 1 & 1 \\
 1 & & 2 \\
 k & 1 & 4415.874
\end{array}
\]

(3.6)

\[
\begin{array}{cccc}
  & 1 & 1 & 2 & 3 & 4 \\
 1 & & & & & \\
 2 & & & & & \\
 3 & & & & & 358.542
\end{array}
\]

(3.7)

\[
\begin{array}{ccc}
  & 1 & 1 \\
 1 & & 2 \\
 k & 1 & 5277.946
\end{array}
\]

(3.8)

\[
\begin{array}{cccc}
  & 1 & 1 & 2 & 3 & 4 \\
 1 & & & & & \\
 2 & & & & & \\
 3 & & & & & 27990.74
\end{array}
\]

(3.10)

\[
\begin{array}{ccc}
  & 1 & 1 \\
 1 & & 2 \\
 k & 1 & 404.69
\end{array}
\]
The bounds for individual singular values are found from the diagonals of these tables.
We have seen that some bounds for \( \alpha_1 = \alpha_2 = \ldots = \alpha_n = 1 \) are better than those for \( \alpha_i \)'s (\( i = 1, \ldots, n \)) such that \( \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2 = n \) while some bounds are worse.

ÖZET

\( A \), singüler değerleri \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \) olan \( n \times n \) tipinde bir kompleks matris ve \( 1 \leq k < l \leq n \) olsun. Bu çalışmada, \( r_i(c_i) \), \( i \)-inci satır Euclidian normu (sütun Euclidian normu) ve \( \alpha_i \) ler, \( \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2 = n \) olacak şekilde pozitif reel sayılar olmak üzere \( \sigma_1 \ldots \sigma_k \), \( \sigma_k \ldots \sigma_l \) ve \( \sigma_{n-k+1} \ldots \sigma_n \) için \( k, n, r_i(c_i), \alpha_i \) ve \( \det A \) içeren sınırlar elde edildi.

REFERENCES


e-mail addresses: ayse2dilek@yahoo.com
dtasci@gazi.edu.tr