ON H-CONTINUITY OF MULTIFUNCTIONS DEFINED FROM A PRODUCT SPACE TO A PRODUCT SPACE

M. AKDAĞ

Cumhuriyet University, Faculty of Arts and Science, Department of Mathematics, 58140 Sivas-TURKEY
E-mail: makdag@cumhuriyet.edu.tr

(Received: Nov. 8, 2002 ; Revised: Feb. 17, 2003 ; Accepted: Feb. 18, 2003 )

ABSTRACT

The purpose of the present paper is to investigate some properties of H-continuous (Almost H-continuous) multifunctions defined from a product space to a product space. The relations between the strongly closed graph of a multivalued functions and H-upper semicontinuity of multivalued functions are also investigated.

1. INTRODUCTION

The H-continuity of single valued function has been studied by P. E. Long and T. R. Hamlett [1,1975]. The study for multivalued functions of H-continuity has been taken over by V. Popa [2] and R. E. Smithson [3]. Moreover, the H-almost upper semi continuity of single valued functions has been extended to multivalued functions by Y. Kütçük and M. Akdağ [4].

In this paper, I studied H-almost upper semicontinuity and H-upper semi continuity and C-upper semicontinuity of multivalued functions defined from a product space to a product space. Also, I obtained some relations between strongly closed graph and H-upper semicontinuity of multivalued functions.

2. PRELIMINARIES

A multivalued function $F:X \rightarrow Y$ is a function $F:X \rightarrow P(Y) \setminus \{\emptyset\}$ where $P(Y)$ is the power set of $Y$. For a multivalued function $F$, the upper and the lower inverse of a set $B$ of $Y$ will be denoted by $F^+ (B)$ and $F^-(B)$, respectively where $F^+ (B)=\{ x\in X | F(x) \subset B \}$ and $F^- (B)=\{ x\in X | F(x) \supset \emptyset \} [5]$. The graph $G(F)$ of a multivalued function $F:X \rightarrow Y$ is the subset $\{(x,y) | x\in X, y\in F(x)\}$ of $X \times Y$. A multivalued function $F:X \rightarrow Y$ has a closed graph (strongly
closed graph) if and only if for each \((x,y)\in X\times Y\setminus G(F)\), there exist open sets \(U\) and \(V\) containing \(x\) and \(y\), respectively such that \((U\times V)\setminus (U\times V)\setminus G(F)\setminus =\emptyset\) [6].

A subset \(A\) of a space \(X\) is a quasi \(H\)-closed set (or \(H\)-set) if for every open cover \(\mathcal{G} = \{U_\alpha \mid \alpha \in \Delta\}\) of \(A\) there exist a finite subcover \(\{U_1, U_2, \ldots, U_n\}\) of \(\mathcal{G}\) such that \(A \subset \bigcup_{i=1}^n \text{cl}(U_i)\) [7]. If \(X\) is an \(H\)-set, then \(X\) is \(H\)-closed [7]. A space \(X\) is \(H\)-closed if it is Hausdorff and \(H\)-set [7]. A Hausdorff space \(X\) is locally \(H\)-closed if for each \(x \in X\), there exists a \(H\)-closed neighbourhood of \(X\) [8]. A space \(X\) is \(C\)-compact if every closed subset of \(X\) is an \(H\)-set [9]. A space \(X\) is \(HC\)-space if every \(H\)-set of \(X\) is a closed set [5].

Let \(X\) and \(Y\) be two topological spaces and \(F\) be a multifunction valued from \(X\) to \(Y\). Then \(F\) is said to be \(H\)-upper semicontinuous (\(H\)-almost upper semicontinuous) at \(x \in X\) if for any \(H\)-set \(V\) with \(F(x) \varsubsetneq V = \emptyset\), there exists a neighbourhood \(U\) of \(x\) such that for \(x \in U\), \(F(x) \cap V = \emptyset\). If for every point \(x\) in \(X\), \(F\) is \(H\)-upper semicontinuous (\(H\)-almost upper semicontinuous), then \(F\) is \(H\)-upper semicontinuous (\(H\)-almost upper semicontinuous) at \(X\) [5].

A multifunction \(F\) is said to be \(C\)-upper semicontinuous at \(x \in X\) if for any compact set \(V\) with \(F(x) \cap V = \emptyset\), there exists a neighbourhood \(U\) of \(x\) such that for \(x \in U\), \(F(x) \varsubsetneq V = \emptyset\) [10].

A multifunction \(F\) is said to be point closed (point compact) if for each \(x \in X\), \(F(x)\) is closed (compact).

### 3. PRODUCT SPACES

Let \((X_\alpha, \tau_\alpha)\) and \((Y_\alpha, h_\alpha)\) be topological spaces, \((\Pi X_\alpha, \tau)\) be product space and also let us denote that \(F(x) = \{F_\alpha(x_\alpha)\}\) such that \(x = \{x_\alpha\}_{\alpha \in \Delta}\), \(F_\alpha : X_\alpha \rightarrow Y_\alpha\) and \(F : \Pi X_\alpha \rightarrow \Pi Y_\alpha\).

#### Lemma 3.1.
Let \(f(X, \tau) \rightarrow (Y, h)\) a continuous function. If \(A\) is a \(H\)-set of \(X\), then \(f(A)\) is \(H\)-set of \(Y\).

**Proof:** Let \(A\) be a \(H\)-set in \(X\) and let \(\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}\) be an open cover of \(f(A)\). Then \(f(A) \subset \bigcup_{\alpha \in \Delta} U_\alpha\) and \(A \subset f^{-1}(f(A)) \subset f^{-1}(\bigcup_{\alpha \in \Delta} U_\alpha) = \bigcup_{\alpha \in \Delta} f^{-1}(U_\alpha)\). Since \(f\) is continuous, \(f^{-1}(U_\alpha)\) is an open set in \(X\) for each \(\alpha \in \Delta\). Thus \(\mathcal{V} = \{f^{-1}(U_\alpha) \mid \alpha \in \Delta\}\) is an open cover of \(A\). Since \(A\) is an \(H\)-set, there exists a finite cover \(\{U_1, U_2, \ldots, U_n\}\) of \(A\) such that

\[
A \subset \bigcup_{i=1}^n f^{-1}(U_i).
\]

Therefore, since \(f\) is continuous,
\[
f(A) \subset \bigcap_{i=1}^{n} f^{-1}(U_i) \subset \bigcup_{i=1}^{n} f^{-1}(U_i) \subset \bigcup_{i=1}^{n} f(f^{-1}(U_i)) \subset \bigcup_{i=1}^{n} f(U_i).
\]

Thus \(f(A)\) is an H-set in \(Y\).

**Lemma 3.2.** If \(f: X \to Y\) is continuous, onto, open and \(X\) is locally H-closed Hausdorff space, then \(Y\) is locally H-closed.

**Proof:** Let \(y \in Y\). Then there is a \(x \in \{f^{-1}(y)\} \subset X\). Since \(X\) is locally H-closed, there exists a quasi H-closed neighbourhood \(K\) of \(x\). Hence there is an open set \(U\) in \(X\) such that \(x \in U \subset K\). Since \(f\) is continuous and open, \(f(K)\) is a quasi H-closed neighbourhood of \(y\) from Lemma 3.1. Thus \(Y\) is locally H-closed.

**Lemma 3.3.** If \(\prod X_\alpha\) is locally H-closed, then \(X_\alpha\) is locally H-closed for each \(\alpha \in \Delta\).

**Proof:** Since the \(\alpha\)-th projection function \(P_\alpha: \prod X_\alpha \to X_\alpha\) is continuous, open and from Lemma 3.2., the proof is clear.

**Lemma 3.4.** For each \(\alpha \in \Delta\), multifunctions \(F_\alpha: X_\alpha \to Y_\alpha\) is point closed if and only if multifunction \(F: \Pi X_\alpha \to \Pi Y_\alpha\) is point closed where \(F(x) = \{F_\alpha(x_\alpha)\}\) for \(x = \{x_\alpha\}\).

**Proof:** Let \(x \in \Pi X_\alpha\) with \(x = \{x_\alpha\}, \alpha \in \Delta\). Then \(F(x) = \{F_\alpha(x_\alpha)\} = \{\overline{F_\alpha(x_\alpha)}\} = \prod F_\alpha(x_\alpha) = \overline{\prod F_\alpha(x_\alpha)} = \overline{F(x)}\)

**Lemma 3.5.** If \(Y_\alpha\) is H-closed, then \(\prod Y_\alpha\) is H-closed for each \(\alpha \in \Delta\) [7].

**Lemma 3.6.** Let \(\{X_\alpha\}_{\alpha \in \Delta}\) and \(\{Y_\alpha\}_{\alpha \in \Delta}\) be two families of topological spaces. \(F_\alpha: X_\alpha \to Y_\alpha\) be a multifunction for each \(\alpha \in \Delta\). If \(F_\alpha\) is strongly closed for each \(\alpha \in \Delta\), then \(G(F)\) which is the graph of \(F\) is strongly closed.

**Proof:** Let \((x,y) \in G(F)\). Then there is a \(\beta \in \Delta\) such that \(y_\beta \notin F_\beta(x_\beta)\). Since \(G(F_\beta)\) is strongly closed, there exist two open sets \(U_\beta\) and \(V_\beta\), respectively in \(X_\beta\) and in \(Y_\beta\) such that \(x_\beta \in U_\beta\) and \(y_\beta \in V_\beta\) and \(F_\beta(U_\beta) \cap V_\beta = \emptyset\). If we take \(U = U_\beta \times \prod_{\alpha \neq \beta} X_\alpha\) and \(V = V_\beta \times \prod_{\alpha \neq \beta} Y_\alpha\), then \(U\) and \(V\) are open sets in \(\Pi X_\alpha\) and \(\Pi Y_\alpha\), respectively and \(x \in U, y \in V\) and \(F(U) \cap V = \emptyset\). Indeed:

\[
F(U) = (F_\beta(U_\beta)) \times \prod_{\alpha \neq \beta} Y_\alpha \quad (\overline{V_\beta} \times \prod_{\alpha \neq \beta} Y_\alpha) = (F_\beta(U_\beta) \times \overline{V_\beta}) \times \prod_{\alpha \neq \beta} Y_\alpha = \emptyset. \text{ Thus } G(F) \text{ is strongly closed.}
\]
Lemma 3.7. If for each $\alpha \in \Delta$, $X_\alpha$ is HC-closed, then $\prod X_\alpha$ is HC-space.

Proof: Let $G = \prod G_\alpha \subset \prod X_\alpha$ be a quasi H-closed set. From Lemma 3.1. and since $\alpha$-th projection $P_\alpha$ is continuous for each $\alpha \in \Delta$, $P_\alpha(G) = G_\alpha \subset X_\alpha$ is a quasi H-closed set. Since $X_\alpha$ is HC-space, $G_\alpha$ is closed in $X_\alpha$ that is $G_\alpha = G_\alpha$. Thus $G = \prod G_\alpha = \prod \overline{G_\alpha} = \prod \overline{G_\alpha} = \overline{G}$ and $\prod X_\alpha$ is HC-space.

Theorem 3.8. Let $\{X_\alpha\}_{\alpha \in \Delta}$ and $\{Y_\alpha\}_{\alpha \in \Delta}$ be two families of topological spaces. If for each $\alpha \in \Delta$, $F_\alpha: X_\alpha \rightarrow Y_\alpha$ is H-u.s.c. (C-u.s.c.), point compact (point closed) and $Y_\alpha$ is locally H-closed Hausdorff space (locally compact Hausdorff) then multifunction $F: \prod X_\alpha \rightarrow \prod Y_\alpha$ is H-u.s.c. (C-u.s.c.).

Proof: Since for each $\alpha \in \Delta$, $F_\alpha$ is H-u.s.c. (C-u.s.c.) from the Propositions 4.11 and 4.12 in [11] and the Proposition 14 in [10], $G(F)$ is strongly closed. From Lemma 3.4., $G(F)$ is strongly closed. Thus $F$ is H-u.s.c. (C-u.s.c.)

Theorem 3.9. If multifunction $F: \prod X_\alpha \rightarrow \prod Y_\alpha$ is H-almost u.s.c., then for each $\alpha \in \Delta$, multifunctions $F_\alpha: X_\alpha \rightarrow Y_\alpha$ is H-almost u.s.c.

Proof: Let $x_\beta \in X_\beta$ for any $\beta \in \Delta$ and let $V_\beta$ be a set such that $F_\beta(x_\beta) \subset V_\beta \subset Y_\beta$ and its complement is quasi H-closed. Also we define a set $A_\beta = \{x \in \prod X_\alpha | \text{the } \beta\text{-th coordinate of } x \text{ is } x_\beta\}$ for $x \in A_\beta$ and since $A_\beta \subset \prod X_\alpha$, $x \in \prod X_\alpha$. On the other hand, using the $\beta$-th projection $P_\beta: \prod Y_\alpha \rightarrow Y_\beta$, we obtain $P^{-1}_\beta(V_\beta) = V_\beta \times \prod_{\alpha \neq \beta} Y_\alpha$.

Since $F$ is H-almost u.s.c., there exists an open set $U \subset \prod X_\alpha$ such that $F(U) \subset \prod_{\alpha \neq \beta} \overline{V_\beta} \times \prod_{\alpha \neq \beta} Y_\alpha$. Since $P_\beta$ is open, $P_\beta(U_\beta) = U_\beta \subset X_\beta$ is an open set in $X_\beta$. Thus $F_\beta(U_\beta) \subset \overline{V_\beta}$ and $F_\beta$ is H-almost u.s.c. at $x_\beta \in X_\beta$. Since $x_\beta$ is an arbitrary point, $F_\beta$ is H-almost u.s.c. on $X_\beta$.

Theorem 3.10. Let for each $\alpha \in \Delta$, $Y_\alpha$ be H-closed, locally H-closed and HC-space. If multifunction $F: \prod X_\alpha \rightarrow \prod Y_\alpha$ is H-u.s.c., then multifunctions $F_\alpha: X_\alpha \rightarrow Y_\alpha$ is H-u.s.c. for each $\alpha \in \Delta$.

Proof: Since $Y_\alpha$ is H-closed, locally H-closed and HC-space from Lemma 3.4., Lemma 3.5. and Lemma 3.7., $\prod_{\alpha \in \Delta} Y_\alpha$ is H-closed, locally H-closed and HC-space.
for each \( \alpha \in \Delta \). Since \( F \) is H-u.s.c., \( F \) is H-almost u.s.c. [4]. Thus for each \( \alpha \in \Delta \), \( F_\alpha \) is H-almost u.s.c. from Lemma 3.7 and from the Proposition 3.13 in [4], \( F_\alpha \) is H-u.s.c.

**Corollary 3.11.** Let for each \( \alpha \in \Delta \), \( Y_\alpha \) be H-closed, locally H-closed and HC-space and let \( F_\alpha \) be compact point. Then multifunction \( F: \prod X_\alpha \rightarrow \prod Y_\alpha \) is H-u.s.c. if and only if for each \( \alpha \in \Delta \), multifunctions \( F_\alpha: X_\alpha \rightarrow Y_\alpha \) is H-u.s.c.

**Proof:** (\( \Rightarrow \)): From Theorem 3.10., it is clear.

(\( \Leftarrow \)): From Theorem 3.8., it is clear.

**Theorem 3.12.** If for each \( \alpha \in \Delta \) multifunctions \( F_\alpha: X \rightarrow Y_\alpha \) has strongly closed graph then multifunction \( F: X \rightarrow Y_\alpha \) has strongly closed graph.

**Proof:** Let \( (x,y) \in G(F) \). Then \( y \in F(x) \) and for at least a \( \beta \in \Delta \), \( y_\beta \not\in F_\beta(x) \). Hence \( (x,y_\beta) \not\in G(F) \). Since \( G(F_\beta) \) is strongly closed, there exist two open sets \( U \) and \( V \) in \( X \) and in \( Y \), respectively such that \( x \in U \), \( y \in V \) and \( F_\beta(U) \supseteq \emptyset \). If we choose \( V = \prod_{\alpha \neq \beta} Y_\alpha \), then \( V \) is open in \( \prod Y_\alpha \) and \( y \in V \) and \( F(U) \cap V = \emptyset \). Indeed:

\[
F(U) \cap V = F(U) \cap \left( \prod_{\alpha \neq \beta} Y_\alpha \right) = \left( \prod_{\alpha \neq \beta} F_\alpha(U) \right) \cap \emptyset = \emptyset.
\]

Thus \( G(F) \) is strongly closed.

**Corollary 3.13.** If \( Y_\alpha \) is locally H-closed Hausdorff (locally compact Hausdorff) and for each \( \alpha \in \Delta \), multifunctions \( F_\alpha: X \rightarrow Y_\alpha \) is compact point (closed point) and H-u.s.c. (C-u.s.c.), then multifunction \( F: X \rightarrow \prod Y_\alpha \), H-u.s.c. (C-u.s.c.).

**Proof.** From Theorem 3.12. and the Proposition 3.13 in [4], it is clear.

**REFERENCES**


