SOME NEW GENERALIZED SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

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ABSTRACT

In this paper, we introduce some new generalized sequence spaces using Orlicz function. We also examine some properties of these sequence space.

1. INTRODUCTION

Let $l_{\infty}$, $\ell$ and $c_0$ be the Banach spaces of bounded, convergent and null sequences $x=(x_k)$, respectively, normed, as usual, by $\|x\| = \sup k |x_k| < \infty$.

Lindenstrauss and Tzafriri [1] used the idea of Orlicz function to construct the sequence space

$$ l_M = \left\{ x : \sum_k M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}. $$

The space $l_M$ with the norm

$$ \|x\| = \inf \left\{ \rho > 0 : \sum_k M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\} $$

becomes a Banach space which is called an Orlicz sequence space.
The space \( l_M \) is closely related to the space \( l_p \) which is an Orlicz sequence space with \( M(x) = x^p, 1 \leq p < \infty \).

In the present note we introduce and examine some properties of four sequence spaces defined by using Orlicz function \( M \), which generalize the well known Orlicz sequence space \( l_M \) and \( l_\infty (p, s) \), \( c(p, s) \) and \( c_0 (p, s) \).

An Orlicz function is a function \( M: [0, \infty[ \to [0, \infty[ \), which is continuous, non-decreasing and convex with \( M(0)=0, M(x) > 0 \) for \( x>0 \) and \( M(x) \to \infty \) as \( x \to \infty \).

2. MAIN RESULTS

Let \( p= (p_k) \) be a sequence of positive real numbers. We define the following sequence spaces

\[
l_M (p, s) = \left\{ x \in w : \sup_n \sum_k k^{-s} \left[ M \left( \frac{|x_{k+n}|}{\rho} \right) \right]^{p_k} < \infty, \quad \text{for some } \rho > 0 \text{ and } s \geq 0 \right\},
\]

\[
l_\infty (M, p, s) = \left\{ x \in w : \sup_n \sum_k k^{-s} \left[ M \left( \frac{|x_{k+n}|}{\rho} \right) \right]^{p_k} < \infty, \quad \text{for some } \rho > 0 \text{ and } s \geq 0 \right\},
\]

\[
c(M, p, s) = \left\{ x \in w : k^{-s} \left[ M \left( \frac{|x_{k+n} - L|}{\rho} \right) \right]^{p_k} \to 0 \text{ as } k \to \infty, \quad \text{for some } \rho, L > 0, \text{ and } s \geq 0, \text{ uniformly in } n \right\},
\]

\[
c_0 (M, p, s) = \left\{ x \in w : k^{-s} \left[ M \left( \frac{|x_{k+n}|}{\rho} \right) \right]^{p_k} \to 0 \text{ as } k \to \infty, \quad \text{for some } \rho > 0, \text{ and } s \geq 0, \text{ uniformly in } n \right\}.
\]

When \( p_k=1 \) for all \( k, n=0 \) and \( s=0 \), then \( l_M (p, s) \) becomes \( l_{M} \). When \( M(x)=x \) and \( s=0 \) then the family of sequences defined above become \( l_{(p)} \), \( l_{\infty (p)} \), \( c(p) \) and \( c_0 (p) \) respectively [2].
When $M(x) = x$ and $n = 0$, then $l_M(p, s)_n$ becomes $l_{(p, s)}$ which has been investigated by Bulut and Çakar [3] and $l_\infty(M, p, s), c(M, p, s)$ and $c_o(M, p, s)$ become $l_\infty(p, s), c(p, s)$ and $c_o(p, s)$ which has been investigated by Başarir [4].

In order to discuss the properties of $l_{M}(p, s)$, we assume that $p = (p_k)$ is bounded.

**Theorem 1.** Let $H = \sup_k p_k < \infty$ then $l_M(p, s)$ is a linear set over the set of complex numbers $C$.

**Proof.** Let $x, y \in l_M(p, s)$ and $\alpha, \beta \in C$. In order to prove the result we need to find some $\rho_3$ and $s \geq 0$ such that

$$\sup_n \sum_k k^{-s} \left[ M \left( \frac{|\alpha x_{k+n} + \beta y_{k+n}|}{\rho_3} \right) \right]^{p_k} < \infty .$$

Since $x, y \in l_M(p, s)$, therefore there exists some positive $\rho_1, \rho_2$ and $s \geq 0$ such that

$$\sup_n \sum_k k^{-s} \left[ M \left( \frac{|x_{k+n}|}{\rho_1} \right) \right]^{p_k} < \infty ,$$

and

$$\sup_n \sum_k k^{-s} \left[ M \left( \frac{|y_{k+n}|}{\rho_2} \right) \right]^{p_k} < \infty .$$

Define $\rho = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $M$ is non-decreasing and convex

$$\sum_k k^{-s} \left[ M \left( \frac{|\alpha x_{k+n} + \beta y_{k+n}|}{\rho_3} \right) \right]^{p_k} \leq \sum_k k^{-s} \left[ M \left( \frac{|\alpha x_{k+n}|}{\rho_3} + \frac{|\beta y_{k+n}|}{\rho_3} \right) \right]^{p_k}$$

$$\leq \sum_k k^{-s} \frac{1}{2^{p_k}} \left[ M \left( \frac{|x_{k+n}|}{\rho_1} + \frac{|y_{k+n}|}{\rho_2} \right) \right]^{p_k}.$$
\[
< \sum_{k} k^{-s} \left[ M \left( \frac{|x_{k+n}|}{\rho_1} \right) + M \left( \frac{|y_{k+n}|}{\rho_2} \right) \right]^{p_k} \\
\leq C \sum_{k} k^{-s} \left[ M \left( \frac{|x_{k+n}|}{\rho_1} \right) \right]^{p_k} + C \sum_{k} k^{-s} \left[ M \left( \frac{|y_{k+n}|}{\rho_2} \right) \right]^{p_k}
\]
for all \( n \),

where \( C = \max (1, 2^{H-1}) \). This proves that \( l_M(p, s) \) is linear.

**Theorem 2.** \( l_M(p, s) \) is paranormed space with the paranorm

\[
G(x) = \inf \left\{ \rho^p_H : \left( \sum_k k^{-s} \left[ M \left( \frac{|x_{k+n}|}{\rho} \right) \right]^{p_k} \right)^{1/H} \leq 1, n = 1, 2, \ldots, s \geq 0 \right\}
\]

where \( H = \max(1, \sup_k p_k) \).

**Proof.** Clearly \( G(x) = G(-x) \). The subadditivity of \( G \) follows from Theorem 1. Since \( M(0) = 0 \), we get \( \inf \left\{ \rho^p_H \right\} = 0 \) for \( x = 0 \). Conversely, suppose that \( G(x) = 0 \). Then it is easy to see that \( x = 0 \). Finally using the same technique of Theorem 2 of Parashar and Choudhary [5], it can be easily seen that scalar multiplication is continuous. This completes the proof.

**Remark.** It can easily be verified that when \( M(x) = x \) and \( n = 0 \), the paranorm defined \( l_M(p, s) \) and paranorm defined in \( l(p, s) \) are the same.

An Orlicz function \( M \) can always be represented by Krasnoselskii and Rutitsky [6] in the following integral form
\[ M(x) = \int_0^x q(t) \, dt \]

where \( q \), known as the kernel of \( M \), is right-differentiable for \( t \geq 0 \), \( q(0) = 0 \), \( q(t) > 0 \) for \( t > 0 \), \( q \) is non-decreasing and \( q(t) \to \infty \) as \( t \to \infty \).

**Theorem 3.** Let \( 1 \leq p_k < \infty \). Then \( \ell_M(p, s) \) is complete paranormed space with

\[
G(x) = \inf \left\{ \rho^p \gamma_H : \left( \sum_k k^{-s} \left[ M\left( \frac{|x_k+n|}{\rho} \right) \right]^{p_k} \right)^{\gamma_H} \leq 1, \, n = 1, 2, ..., s \geq 0 \right\}.
\]

**Proof.** The proof follows on the same lines as adopted by Parashar and Choudhary [5, Theorem 3]. So we omit it.

**Theorem 4**

(i). Let \( 0 < p_k \leq q_k < \infty \) for each \( k \). Then \( \ell_M(p, s) \subset \ell_M(q, s) \).

(ii). \( s_1 \leq s_2 \) implies \( \ell_M(p, s_1) \subset \ell_M(p, s_2) \).

**Proof (i).** Let \( x \in \ell_M(p, s) \). Then there exists some \( \rho > 0 \) and \( s \geq 0 \) such that

\[
\sup_n \sum_k k^{-s} \left[ M\left( \frac{|x_k+n|}{\rho} \right) \right]^{p_k} < \infty.
\]

This implies that

\[
i^{-s} \left[ M\left( \frac{|x+n|}{\rho} \right) \right] \leq 1 \quad \text{for sufficiently large values of } i \text{ and all } n.
\]

Since \( M \) is non-decreasing, we get
\[ \sum_{k} k^{-s} \left[ M \left( \frac{|x_{k+n}|}{\rho} \right) \right]^q \leq \sum_{k} k^{-s} \left[ M \left( \frac{|x_{k+n}|}{\rho} \right) \right]^p < \infty. \]

Thus we get \( x \in L_M(q,s) \).

(ii). Let \( s_1 \leq s_2 \). Then \( k^{-s_2} \leq k^{-s_1} \) for all \( k \). Since

\[ k^{-s_2} \left[ M \left( \frac{|x_{k+n}|}{\rho} \right) \right]^p \leq k^{-s_1} \left[ M \left( \frac{|x_{k+n}|}{\rho} \right) \right]^p \text{ for all } n, \]

and then

\[ \sum_{k} k^{-s_2} \left[ M \left( \frac{|x_{k+n}|}{\rho} \right) \right]^p \leq \sum_{k} k^{-s_1} \left[ M \left( \frac{|x_{k+n}|}{\rho} \right) \right]^p \]

this inequality implies that \( L_M(p,s_1) \subset L_M(p,s_2) \).

**Definition.** (Krasnoselskii and Rutitsky [6]) An Orlicz function is said to satisfy \( \Delta_2 \)-condition for all values of \( u \), if there exists a constant \( K > 0 \), such that

\[ M(2u) \leq K M(u), \quad u \geq 0. \]

The \( \Delta_2 \)-condition is equivalent to the satisfaction of the inequality

\[ M(Lu) \leq KLM(u) \]

for all values of \( u \) and for \( L > 1 \).

**Theorem 5.** Let \( M \) be an Orlicz function which satisfies \( \Delta_2 \)-condition. Then

(i). \( L_\infty \subset L_M(p,s) \),

(ii). \( L(p,s) \subset L_M(p,s) \).

**Proof** (i). Let \( x \in L_\infty \). This implies that \( |x_{k+n}| \leq N \) for all \( k \) and \( n \). So that
\[ k^{-s} \left[ M \left( \frac{|x_{k+n}|}{\rho} \right) \right]^{p_k} \leq k^{-s} \left[ M \left( \frac{N}{\rho} \right) \right]^{p_k} \leq k^{-s} [KLM(N)]^H \text{ by } \Delta_2 \]

condition, where \( H = \max(1, \sup_k p_k) \). Hence

\[ \sum_k k^{-s} \left[ M \left( \frac{|x_{k+n}|}{\rho} \right) \right]^{p_k} < \infty . \]

This shows that \( l_{\infty} \subset l_M(p,s) \).

(ii). Using the same technique of Theorem 3 by Esi [8] it is easy to prove the Theorem.

Now we investigate some properties of spaces \( c_0 (M,p,s) \), \( c(M,p,s) \) and \( l_{\infty} (M,p,s) \) defined earlier. We first state simple property of these spaces.

**Theorem 6.** Let \( p = (p_k) \) be bounded. Then \( c_0 (M,p,s) \), \( c(M,p,s) \) and \( l_{\infty} (M,p,s) \) are linear spaces.

**Proof.** Omitted.

**Theorem 7.** Let \( \sup_k p_k = H < \infty \). Then \( c_0 (M,p,s) \) is a linear topological space paranormed by

\[ g(x) = \inf \left\{ \rho / H : \left[ \rho^{-s} \left( M \left( \frac{|x_{k+n}|}{\rho} \right) \right]^{p_k} \right]^{1/H} \leq 1, \quad n = 1,2,..., s \geq 0 \right\} . \]

**Proof.** Omitted.

**Theorem 8**

(i). Let \( 0 < \inf p_k \leq p_k \leq 1 \). Then \( c_0 (M,p,s) \subset c_0 (M,s) \), \( c(M,p,s) \subset c(M,s) \) and \( l_{\infty} (M,p,s) \subset l_{\infty} (M,s) \).

(ii). Let \( 1 \leq p_k \leq \sup_k p_k < \infty \). Then \( c_0 (M,s) \subset c_0 (M,p,s) \), \( c(M,s) \subset c(M,p,s) \), and \( l_{\infty} (M,s) \subset l_{\infty} (M,p,s) \).

**Proof (i).** Let \( x \in c_0 (M,p,s) \). Since \( 0 < \inf p_k \leq 1 \), we get
\( k^{-s} \left[ M \left( \frac{|X_{k+n}|}{\rho} \right) \right] \leq k^{-s} \left[ M \left( \frac{|X_{k+n}|}{\rho} \right) \right]^{p_k} \) for all \( n \),

and hence \( x \in c_0(M,s) \).

(ii). Let \( 1 \leq p_k \leq \sup p_k < \infty \) for each \( k \) and \( x \in c_0(M,s) \). Then for each \( 0 < \varepsilon < 1 \) and for all \( n \), there exists a positive integer \( N \) such that

\[
k^{-s} \left[ M \left( \frac{|X_{k+n}|}{\rho} \right) \right] \leq \varepsilon < 1
\]

for all \( k \geq N \) and \( s \geq 0 \). This implies that

\[
k^{-s} \left[ M \left( \frac{|X_{k+n}|}{\rho} \right) \right]^{p_k} \leq k^{-s} \left[ M \left( \frac{|X_{k+n}|}{\rho} \right) \right].
\]

Thus we get \( x \in c_0(M,p,s) \).

The other inclusions can be treated similarly.

**Theorem 9**

(i). Let \( 0 < p_k \leq q_k < \infty \) and \( \frac{q_k}{p_k} \) be bounded. Then \( c_0(M,q,s) \subset c_0(M,p,s) \) and \( c(M,q,s) \subset c(M,p,s) \).

(ii). Let \( M \) be an Orlicz function which satisfies \( \Delta_2 \)-condition. Then \( c_0(p,s) \subset c_0(M,p,s) \), \( c(p,s) \subset c(M,p,s) \) and \( l_\infty(p,s) \subset l_\infty(M,p,s) \).

(iii). \( c(M_1,p,s) \cap c(M_2,p,s) \subset c(M_1+M_2,p,s) \)

where \( M_1 \) and \( M_2 \) are two Orlicz functions and \( s \geq 0 \).

(iv). \( s_1 \leq s_2 \) implies \( c(M,p,s_1) \subset c(M,p,s_2) \).

**Proof (i).** If we take

\[
t_k = k^{-s} \left[ M \left( \frac{|X_{k+n}|}{\rho} \right) \right]^{p_k}
\]
for all $n$, $k$ and $s \geq 0$, then using the same technique of Theorem 2 of Nanda [7], it is easy to prove (i).

(ii) Proof is similar to Theorem 5 (ii).

Using the same technique of Theorem 3 by Esi [9], it is easy to prove Theorem 9(iii) and (iv)

REFERENCES


