ON THE HADAMARD PRODUCTS OF GCD AND LCM MATRICES

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ABSTRACT

Let \( S = \{x_1, x_2, \ldots, x_n\} \) be a set of distinct positive integers. The matrix \((S)\) having the greatest common divisor \((x_i, x_j)\) of \(x_i\) and \(x_j\) as its \(i, j\)-entry is called the greatest common divisor (GCD) matrix on \(S\). The matrix \([S]\) having the least common multiple \([x_i, x_j]\) of \(x_i\) and \(x_j\) as its \(i, j\)-entry is called the least common multiple (LCM) matrix on \(S\). In this paper we obtain some results related with Hadamard products of GCD and LCM matrices. The set \(S\) is factor-closed if it contains every divisor of each of its elements. It is well-known, that if \(S\) is factor-closed, then there exist the inverses of the GCD and LCM matrices on \(S\). So we conjecture that if the set \(S\) is factor-closed, then \((S)\circ(S)^{-1}\) and \([S]\circ[S]^{-1}\) matrices are doubly stochastic matrices and \(\text{tr}\left((S)\circ(S)^{-1}\right) = \text{tr}\left((S)\right) = \sum_{i=1}^{n} x_i\).

1. INTRODUCTION

Let \( S = \{x_1, x_2, \ldots, x_n\} \) be a set of distinct positive integers. The matrix \((S)\) having the greatest common divisor \((x_i, x_j)\) of \(x_i\) and \(x_j\) as its \(i, j\)-entry is called the greatest common divisor (GCD) matrix on \(S\). The study of GCD matrices was introduced by Beslin and Ligh [1]. They have shown that every GCD matrix is positive definite.

The matrix \([S]\) having the least common multiple \([x_i, x_j]\) of \(x_i\) and \(x_j\) as its \(i, j\)-entry is called the least common multiple (LCM) matrix on \(S\). Smith [3] also considered the determinant of LCM matrix on a factor-closed set. We note that GCD matrix \((S)\) and LCM matrix \([S]\) are invertible when \(S\) is factor-closed set. In this
paper we obtained some results related with determinant, rank, norm and permanent of the Hadamard product of GCD matrix $\alpha(S)$ and LCM matrix $\gamma(S)$. Moreover we conjecture that if $S$ is factor-closed, then the Hadamard product of GCD matrix $(S)$ and the inverse of GCD matrix $(S)^{-1}$ is a doubly stochastic matrix and also the Hadamard product of LCM matrix $\gamma(S)$ and the inverse of LCM matrix $\gamma(S)^{-1}$ is a doubly stochastic matrix. Again we conjecture that if $S$ is factor-closed, then

$$\text{tr}(\alpha(S)\gamma(S)^{-1}) = \text{tr}(\alpha(S)) = \sum_{i=1}^{n} x_i.$$ 

2. MAIN RESULTS

Definition 1. The Hadamard product of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size is just their element-wise product $A \odot B = (a_{ij}b_{ij}).$

Lemma 1. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. If $(S)$ is the GCD matrix and $\gamma(S)$ is the LCM matrix defined on $S$, then

$$((S)\gamma(S))_{ij} = \begin{cases} x_i^2 & \text{if } i = j \\ x_i x_j & \text{if } i \neq j \end{cases}.$$

Proof. Consider the set $S$ and if we denote the greatest common divisor of $x_i$ and $x_j$ with $(x_i, x_j)$ and the least common multiple of $x_i$ and $x_j$ with $[x_i, x_j]$, then we have

$$(x_i, x_j)[x_i, x_j] = x_i x_j, \ (i, j = 1,2,\ldots,n).$$

Therefore by the definition 1, we get the proof.

Remark 1. Clearly $(S)\gamma(S)$ matrix is symmetric.

Theorem 1. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. If $(S)$ is the GCD matrix and $\gamma(S)$ is the LCM matrix defined on $S$, then
\[ \det((S)\circ[S]) = 0. \]

**Proof.** By Lemma 1 and using the properties of determinants we have

\[
\det((S)\circ[S]) = \det \begin{bmatrix} x_1^2 & x_1 x_2 & \ldots & x_1 x_n \\ x_2 x_1 & x_2^2 & \ldots & x_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n x_1 & x_n x_2 & \ldots & x_n^2 \end{bmatrix}
\]

\[
= (x_1 x_2 \ldots x_n) \det \begin{bmatrix} x_1 & x_2 & \ldots & x_n \\ x_1 & x_2 & \ldots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \ldots & x_n \end{bmatrix}
\]

\[
= (x_1 x_2 \ldots x_n) \cdot 0
\]

\[= 0\]

and thus the proof is complete.

**Definition 2.** Let \( S = \{x_1, x_2, \ldots, x_n\} \) be a set of distinct positive integers. The matrix \( 1/(S) \) is the \( n \times n \) matrix whose \( i, j \)-entry is \( \frac{1}{\gcd(x_i, x_j)} \). We call \( 1/(S) \) the reciprocal GCD matrix on \( S \).

**Corollary 1.** Let \( S = \{x_1, x_2, \ldots, x_n\} \) be a set of distinct positive integers. If \( (S) \) is the GCD matrix and \( 1/(S) \) is the reciprocal GCD matrix defined on \( S \), then

\[ \det((S)\circ 1/(S)) = 0. \]

**Proof.** Since \((S)\circ 1/(S)\) is the matrix whose all entries are equal to 1, it is easily seen that

\[ \det((S)\circ 1/(S)) = 0. \]
Theorem 2. If $(S)$ and $[S]$ are the GCD and LCM matrices defined on $S = \{x_1, x_2, \ldots, x_n\}$ the set of distinct positive integers, respectively, then

$$\text{rank}((S)o[S]) = 1.$$ 

Proof. Firstly by Theorem 1 we say that $\text{rank}((S)o[S]) < n$. If we denote the rows of $(S)o[S]$ with $r_i$ $(i=1,2,\ldots,n)$, then we can write

$$r_1 = x_1(x_1, x_2, \ldots, x_n)$$
$$r_2 = x_2(x_1, x_2, \ldots, x_n)$$
$$\ldots \ldots \ldots$$
$$r_n = x_n(x_1, x_2, \ldots, x_n).$$

So we write

$$r_1 = \frac{x_1}{x_{i+1}}r_{i+1} \quad i = 1,2,\ldots,n-1.$$ 

Therefore $r_1$ is a multiple of the rows $r_2, r_3, \ldots, r_n$. Hence by the elementary row operations it follows that the number of row which isn't zero of the matrix $(S)o[S]$ is 1 and thus the proof is complete.

Definition 3. Let $A = (a_{ij})$ be $n \times n$ matrix over any commutative ring. The permanent of $A$ written $\operatorname{per}(A)$, or simply $\operatorname{Per}A$, is defined by

$$\operatorname{per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{\sigma(i)} ,$$

where the summation extends over all one-to-one functions from $\{1,2,\ldots,n\}$ to $\{1,2,\ldots,n\}$.

Now we can state the following result.
Theorem 3. Let \( S = \{x_1, x_2, \ldots, x_n\} \) be an ordered set of distinct positive integers. If \((S)\) and \([S]\) are the GCD and LCM matrices defined on \(S\), respectively, then

\[
\text{per}((S)\circ[S]) = n! \prod_{i=1}^{n} x_i^2.
\]

Proof. For the brevity if we take \( C = (S)\circ[S] \) and \( C = (c_{ij}) \) then by the Definition 3, we have

\[
\text{per}(C) = \sum_{\sigma} c_{1\sigma(1)}c_{2\sigma(2)}\cdots c_{n\sigma(n)}.
\]

On the other hand we note that the sequence \( (c_{1\sigma(1)}, c_{2\sigma(2)}, \ldots, c_{n\sigma(n)}) \) is called a diagonal of \(C\), and the product \( c_{1\sigma(1)}c_{2\sigma(2)}\cdots c_{n\sigma(n)} \) is a diagonal product of \(C\). Thus the permanent of \(C\) is the sum of all diagonal products of \(C\). Considering the structure of the matrix \( C = (S)\circ[S] \) we get result.

Corollary 2. If \((S)\) is the GCD matrix defined on the set \(S\) of distinct positive integers, and \(1/(S)\) is the reciprocal matrix of GCD matrix, then

\[
\text{per}((S)\circ1/(S)) = n!
\]

Proof. Since \((S)\circ1/(S)\) is the matrix whose all entries are equal to 1, it is easily seen that

\[
\text{per}((S)\circ1/(S)) = n!
\]

Definition 4.(i) The \(1\) norm is defined for \( A \in M_n \) by

\[
\|A\|_1 = \sum_{i,j=1}^{n} |a_{ij}|.
\]

(ii) The Euclidean norm or \(2\) norm is defined for \( A \in M_n \) by
\[ \|A\|_2 = \left( \sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}. \]

(iii) The \( \|A\|_\infty \) norm is defined for \( A \in M_n \) by
\[ \|A\|_\infty = \max_{1 \leq i,j \leq n} |a_{ij}|. \]

(iv) The maximum row sum matrix norm is defined \( A \in M_n \) by
\[ \|A\|_r = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|. \]

(v) The maximum column sum matrix norm is defined \( A \in M_n \) by
\[ \|A\|_c = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|. \]

**Theorem 4.** Let \( S = \{x_1, x_2, \ldots, x_n\} \) be an ordered set of distinct positive integers. If \( (S) \) and \([S]\) are the GCD and LCM matrices defined on \( S \), respectively, then the following statements are satisfied:

(i) \( \| (S)\circ[S] \|_1 = \left( \sum_{i=1}^{n} x_i \right)^2 \)

(ii) \( \| (S)\circ[S] \|_2 = \sum_{i=1}^{n} x_i^2 \)

(iii) \( \| (S)\circ[S] \|_\infty = x_n^2 \)

(iv) \( \| (S)\circ[S] \|_r = \| (S)\circ[S] \|_c = x_n \left( \sum_{i=1}^{n} x_i \right) \).
Proof. If we denote i-th row sum of $(S) o [S]$ with $r_{S_i}$ (i=1,2,...,n). Then we can write

$$r_{S_1} = x_1 \left( \sum_{i=1}^{n} x_i \right)$$

$$r_{S_2} = x_2 \left( \sum_{i=1}^{n} x_i \right)$$

$$
\vdots
$$

$$r_{S_n} = x_n \left( \sum_{i=1}^{n} x_i \right).$$

So we have

$$\left\| (S) o [S] \right\|_1 = r_{S_1} + r_{S_2} + ... + r_{S_n}$$

$$= x_1 \left( \sum_{i=1}^{n} x_i \right) + x_2 \left( \sum_{i=1}^{n} x_i \right) + ... + x_n \left( \sum_{i=1}^{n} x_i \right)$$

$$= (x_1 + x_2 + ... + x_n) \left( \sum_{i=1}^{n} x_i \right)$$

$$= \left( \sum_{i=1}^{n} x_i \right)^2.$$

(ii) Again for the brevity let take as $C = (S) o [S]$ and $C = (c_{ij})$. Then we have

$$\text{tr}(CC^T) = \sum_{i,j=1}^{n} |c_{ij}|^2 = \|C\|_2^2.$$

On the other hand from the structure of the matrix $(S) o [S]$, it is easily seen that
\[ \text{tr}(CC^T) = \left( \sum_{i,j=1}^{n} x_{i,j}^2 \right)^2. \]

So we obtain

\[ \|C\|_2 = \sum_{i=1}^{n} x_i^2. \]

(iii) Since \( S = \{x_1, x_2, \ldots, x_n\} \) is an ordered set of distinct positive integers, without losing the generality we can assume that \( x_1 < x_2 < \ldots < x_n \).

Considering the definition of \( \infty \) norm we find

\[ \| (S) o [S] \|_\infty = x_n^2. \]

(iv) Clearly since \( (S) o [S] \) is symmetric, we have

\[ \| (S) o [S] \|_r = \| (S) o [S] \|_c. \]

On the other hand we can assume that \( x_1 < x_2 < \ldots < x_n \). Therefore by the definition maximum row sum matrix norm (or maximum column sum matrix norm) we have

\[ \| (S) o [S] \|_r = \| (S) o [S] \|_c = x_n \left( \sum_{i=1}^{n} x_i \right). \]

Thus the Theorem 4 is completely proved.

**Definition 5.** A set \( S = \{x_1, x_2, \ldots, x_n\} \) of positive integers is said to be factor-closed (FC) is whenever \( x_i \) is in \( S \) and \( d \) divides \( x_i \), then \( d \) is in \( S \).

The above definition is due to J.J. Malone.
Remark 2. We note that if the set \( S = \{x_1, x_2, \ldots, x_n\} \) of distinct positive integers is not factor-closed, then an LCM matrix may not be invertible. But the GCD matrix \((S)\) defined on any set \( S \) of distinct positive integers is always invertible.

**Theorem 5.** [2] Let \( S = \{x_1, x_2, \ldots, x_n\} \) be a set of distinct positive integers. If \( S \) is factor-closed, then the inverse of the GCD matrix \((S)\) defined on \( S \) is the matrix \((S)^{-1} = (t_{ij})\), where

\[
t_{ij} = \sum_{\substack{x_i \mid x_k \quad \phi(x_k) \quad \mu(x_k / x_i) \mu(x_k / x_j) \quad \mu(x_k / x_j)}} \frac{1}{\phi(x_k)}(x_i, x_j) \sum_{\substack{x_k \mid x_i \quad \phi(x_k) \quad \mu(x_k / x_i) \mu(x_k / x_j) \quad \mu(x_k / x_j)}} \frac{1}{\phi(x_k)}
\]

\( \phi(.) \) is Euler’s totient function and \( \mu(.) \) denotes Moebius function.

Now we present the following.

**Conjecture 1.** Let \( S = \{x_1, x_2, \ldots, x_n\} \) be a set of distinct positive integers and \((S)^{-1} \) be the inverse of the GCD matrix \((S)\) defined on \( S \). If \( S \) is factor-closed, then \((S)\alpha(S)^{-1}\) is a doubly stochastic matrix, i.e.,

\[
\sum_{i=1}^{n} \left( (x_i, x_j) \sum_{\substack{x_i \mid x_k \quad \phi(x_k) \quad \mu(x_k / x_i) \mu(x_k / x_j) \quad \mu(x_k / x_j)}} \frac{1}{\phi(x_k)} \right) = 1 \quad (i = 1, 2, \ldots, n)
\]

and

\[
\sum_{i=1}^{n} \left( (x_i, x_j) \sum_{\substack{x_j \mid x_k \quad \phi(x_k) \quad \mu(x_k / x_i) \mu(x_k / x_j) \quad \mu(x_k / x_j)}} \frac{1}{\phi(x_k)} \right) = 1 \quad (j = 1, 2, \ldots, n).
\]
Theorem 6. [2] Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. If $S$ is factor-closed, then the inverse of the LCM matrix $[S]$ defined on $S$ is the matrix

$$b_{ij} = \frac{1}{x_i x_j} \sum_{\substack{x_k | x_i \\ x_j | x_k}} \frac{1}{g(x_k)} \mu(x_k / x_i) \mu(x_k / x_j),$$

and $g$ is defined for each positive integer $m$ by

$$g(m) = \frac{1}{m} \sum_{d|m} d \mu(d).$$

Conjecture 2. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and $[S]^{-1}$ be the inverse of the LCM matrix $[S]$ defined on $S$. If $S$ is factor-closed, then $[S]o[S]^{-1}$ is a doubly stochastic matrix, i.e.,

$$\sum_{j=1}^{n} \left[ \frac{x_i x_j}{x_i x_j} \sum_{\substack{x_k | x_i \\ x_j | x_k}} \frac{1}{g(x_k)} \mu(x_k / x_i) \mu(x_k / x_j) \right] = 1 \quad (i = 1, 2, \ldots, n)$$

and

$$\sum_{i=1}^{n} \left[ \frac{x_i x_j}{x_i x_j} \sum_{\substack{x_k | x_i \\ x_j | x_k}} \frac{1}{g(x_k)} \mu(x_k / x_i) \mu(x_k / x_j) \right] = 1 \quad (j = 1, 2, \ldots, n).$$

Conjecture 3. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and $(S)^{-1}$ be the inverse of the GCD matrix $(S)$ defined on $S$. If $S$ is factor-closed, then

$$\text{tr}((S)o(S)^{-1}) = \text{tr}((S)) = \sum_{i=1}^{n} x_i,$$

i.e.,
\[
\sum_{i=1}^{n} \left\{ x_i \sum_{x_j|x_k} \frac{1}{\varphi(x_k)} \mu^2 (x_k / x_i) \right\} = \sum_{i=1}^{n} x_i.
\]

**Remark 3.** The Conjecture 3 is not true for \([S]o[S]^{-1}\) matrix, i.e.,

\[
\text{tr}\left([S]o[S]^{-1}\right) \neq \text{tr}([S]) = \sum_{i=1}^{n} x_i.
\]

For example, if \(S = \{1, 2, 4\}\) (we note that the set \(S\) is factor-closed), then we have the following:

\[
[S] = \begin{bmatrix}
1 & 2 & 4 \\
2 & 2 & 4 \\
4 & 4 & 4
\end{bmatrix}, \quad [S]^{-1} = \begin{bmatrix}
-1 & 1 & 0 \\
1 & -3 & 1 \\
0 & 1 & -1
\end{bmatrix}
\]

and

\[
[S]o[S]^{-1} = \begin{bmatrix}
-1 & 2 & 0 \\
2 & -3 & 2 \\
0 & 2 & -1
\end{bmatrix}.
\]

Thus since \(\text{tr}\left([S]o[S]^{-1}\right) = -5\) and \(\text{tr}([S]) = \sum_{i=1}^{n} x_i = 7\), it follows that

\[
\text{tr}\left([S]o[S]^{-1}\right) \neq \text{tr}([S]).
\]
REFERENCES

