HOMOLOGY ACTION ON REGULAR HYPERMAPS OF GENUS 2

M. KAZAZ

Department of Mathematics, Faculty of Science, Celal Bayar University, Manisa, TURKEY

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ABSTRACT

In this paper, we find the homology representations and characters of the orientation-preserving automorphism groups of regular hypermaps of genus 2.

KEYWORDS

Homology Representation, Homology Character, Regular Map, Regular Hypermap.

1. INTRODUCTION

A classification of regular hypermaps of genus 2 have been completed by Azevedo and Jones in [2]. They have shown that there are 43 of them, of which 10 are maps. Another 20 regular hypermaps can be obtained from these regular maps. The remaining 13 regular hypermaps are not associates of these maps: they are 5 regular hypermaps and their associates.

It follows from the result obtained by Singerman in [22] that there are three possible lattices of triangle groups corresponding to genus 2 hypermaps, so that the orientation-preserving automorphism groups Aut H, one for each of the three Riemann surfaces which admit regular hypermap of genus 2. Thus we consider three lattices separately in our calculations. If we choose a homology basis on the hypermap we are let to a matrix representation \( \rho_1 \) under the action of the automorphism group on this homology basis. Then we can find corresponding homology character \( \tau_1 \) of \( G \) which is faithful character.

In [2] Azevedo and Jones showed that all hypermaps of genus 2 are reflexible, that is, each has an additional orientation-reversing automorphism \( \alpha \). This allows us to choose a reflection line on the hypermap, so we obtain a matrix representing \( \alpha \) with respect to a chosen homology basis on the hypermap. One can use these

* This work is originated from the Author’s Ph.D. thesis.
representations and characters to work on finite abelian coverings and reflexivity of regular hypermaps of genus 2.

2. REGULAR HYPERMAPS OF GENUS 2

In this chapter, we will give a brief summary of the regular hypermaps of genus 2. The regular hypermaps on the sphere and torus have been determined by Corn and Singerman in [5]; in each case there are infinitely many regular hypermaps. On the other hand, it is known that the number of regular hypermaps of genus $g \geq 2$ is finite [15]. Moreover, all regular hypermaps $\mathcal{H}$ of genus 2 are known. If $\mathcal{H}$ is a map then the possibilities for its type $\{m, n\}$ and automorphism group $\text{Aut} \mathcal{H}$ are given by Coxeter and Moser in [6, p.140]. If $\mathcal{H}$ is not a map then the possibilities for its type $\{l, m, n\}$ and automorphism group $\text{Aut} \mathcal{H}$ are given by Corn and Singerman in [5]. In fact, Azevedo and Jones in [2], have completed full results by enumerating, describing and constructing all these hypermaps and specifying their full automorphism groups $\text{Aut}^* \mathcal{H}$ (including orientation-reversing automorphisms).

Table 1 shows the regular maps $M_0, M_1, M_2, M_3, M_4, M_5$ of genus 2 together with the duals of $M_1, M_3, M_4$ and $M_5$ which are denoted by $M_j^{(2)}, j = 1, 3, 4, 5$ (to indicate a transposition of vertices and faces) [2].

The entries in each of the six rows are explained as follows: the first two columns give our notation for $M$ and that in [6, Table 9]. The third column gives the type $\{l, m, n\}$ of $M$ as a hypermap. The next column gives the number $\sigma$ of non-isomorphic associates $M^\pi$ of $M$ ($\pi \in S_3$, Machi’s group of hypermap operations which transforms one hypermap $\mathcal{H}$ to another, called an associate $\mathcal{H}^\pi$ of $\mathcal{H}$). The fifth column gives the number $n_i$ ($i = 0, 1, 2$) of hypervertices, hyperedges and hyperfaces, respectively. The sixth column describes the orientation-preserving automorphism group $\text{Aut} M$, and the final column gives the full automorphism group $\text{Aut}^* M$ of order 2 $|\text{Aut} M |$, since each hypermap $M$ is reflexible.
Table 1: The regular maps of genus 2.

<table>
<thead>
<tr>
<th>Map</th>
<th>Type</th>
<th>σ</th>
<th>$n_0$</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>Aut $M$</th>
<th>Aut $^*M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_0$</td>
<td>{8, 8}</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>$C_8$</td>
<td>$D_8$</td>
</tr>
<tr>
<td>$M_1$</td>
<td>{10, 5}</td>
<td>5 2</td>
<td>2 5</td>
<td>1 6</td>
<td></td>
<td>$C_{10}$</td>
<td>$D_{10}$</td>
</tr>
<tr>
<td>$M_2$</td>
<td>{6, 6}</td>
<td>6 2</td>
<td>2 6</td>
<td>1 5</td>
<td></td>
<td>$C_6 \times C_2$</td>
<td>$D_6 \times C_2$</td>
</tr>
<tr>
<td>$M_3$</td>
<td>{4, 8}</td>
<td>4 2</td>
<td>4 8</td>
<td>1 2</td>
<td></td>
<td>$\langle -2, 4</td>
<td>2 \rangle$</td>
</tr>
<tr>
<td>$M_4$</td>
<td>{6, 4}</td>
<td>4 2</td>
<td>6 12</td>
<td>1 4</td>
<td></td>
<td>(4, 6 &gt;2,2)</td>
<td>$D_1 \times D_4$</td>
</tr>
<tr>
<td>$M_5$</td>
<td>{4+4, 3}</td>
<td>3 2</td>
<td>16 24</td>
<td>1 4</td>
<td></td>
<td>$GL_2(3)$</td>
<td>$GL_2(3) \Box C_2$</td>
</tr>
</tbody>
</table>

The regular maps $M_0$, $M_1$, $M_2$, $M_3$, $M_4$, $M_5$ are illustrated in Figure 1.

Figure 1: The regular maps of genus 2.
In each case we have indicated a pair of sides to be identified; the remaining identifications can be deduce by symmetry about the centre, since the maps are regular. For a more detailed account of these hypermaps, their orientation-preserving automorphism and full automorphism groups, see [2].

The remaining 13 regular hypermaps are described in Table 2, each row describes a representative \( H_r(r = 1, \ldots, 5) \) of an \( S_3 \)-orbits of lengths \( \sigma \).

<table>
<thead>
<tr>
<th>Hypermap</th>
<th>Type</th>
<th>( \sigma )</th>
<th>( n_0 ) ( n_1 ) ( n_2 )</th>
<th>( \text{Aut} H )</th>
<th>( \text{Aut}^* H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_1 )</td>
<td>5 5 5</td>
<td>3</td>
<td>1 1 1</td>
<td>( C_5 )</td>
<td>( D_5 )</td>
</tr>
<tr>
<td>( H_2 )</td>
<td>6 6 3</td>
<td>3</td>
<td>1 1 2</td>
<td>( C_6 )</td>
<td>( D_6 )</td>
</tr>
<tr>
<td>( H_3 )</td>
<td>4 4 4</td>
<td>1</td>
<td>2 2 2</td>
<td>( Q_8 )</td>
<td>( Q_8 \cdot C_4 )</td>
</tr>
<tr>
<td>( H_4 )</td>
<td>4 4 3</td>
<td>3</td>
<td>3 3 4</td>
<td>( \hat{D}_3 )</td>
<td>( (4,6 \mid 2,2) )</td>
</tr>
<tr>
<td>( H_5 )</td>
<td>3 3 4</td>
<td>3</td>
<td>8 8 6</td>
<td>( \text{SL}_2(3) )</td>
<td>( \text{GL}_2(3) )</td>
</tr>
</tbody>
</table>

Table 2: The regular hypermaps of genus 2.

We note that in the table, \( Q_8 \cdot C_4 \) shows a central product of \( Q_8 \) by \( C_4 \). The hypermaps \( H_1, \ldots, H_5 \) are illustrated in the following figure, Figure 2.

![Figure 2: The regular hypermaps of genus 2.](image)

Here we use the James model [13] of a hypermap: this is a trivalent graph imbedded in a surface, with hypervertices, hyperedges and hyperfaces represented by the regions labelled 0, 1 and 2 respectively. By contracting each edge separating
hypervertices and hyperedges to a point, we obtain the Cori model [5] of a topological hypermap.

3. INCLUSIONS OF AUTOMORPHISM GROUPS OF GENUS 2

In [22] Singerman gave all possible inclusions \( \Delta_0 \leq \Delta_1 \) between pairs of triangle groups. If \( \Delta_0 \) is normal in \( \Delta_1 \), then Table 3 gives all possible inclusions between hyperbolic triangle groups (they can arise for genus 2 regular hypermaps).

<table>
<thead>
<tr>
<th>( \Delta_1 )</th>
<th>( \Delta_0 )</th>
<th>( [\Delta_1 : \Delta_0] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta(3, 3, t) ), ( t \geq 4 )</td>
<td>( \Delta(t, t, t) )</td>
<td>3</td>
</tr>
<tr>
<td>( \Delta(2, 3, 2t) ), ( t \geq 4 )</td>
<td>( \Delta(t, t, t) )</td>
<td>6</td>
</tr>
<tr>
<td>( \Delta(2, t, 2u) ), ( t \geq 4, t + u \geq 7 )</td>
<td>( \Delta(t, t, u) )</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 3: \( \Delta_0 \leq \Delta_1 \).

If \( \Delta_0 \) is not normal in \( \Delta_1 \), then it follows from the results of Singerman that there are 3 cases which can arise for genus 2 regular hypermaps, these cases are given in Table 4:

<table>
<thead>
<tr>
<th>( \Delta_1 )</th>
<th>( \Delta_0 )</th>
<th>( [\Delta_1 : \Delta_0] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta(2, 3, 4n) ), ( n \geq 2 )</td>
<td>( \Delta(n, 4n, 4n) )</td>
<td>5</td>
</tr>
<tr>
<td>( \Delta(2, 4, 2n) ), ( n \geq 3 )</td>
<td>( \Delta(n, 2n, 2n) )</td>
<td>4</td>
</tr>
<tr>
<td>( \Delta(2, 3, 2n) ), ( n \geq 4 )</td>
<td>( \Delta(2, n, 2n) )</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 4: \( \Delta_0 \leq \Delta_1 \).

Now if \( \Delta_0 \leq \Delta_1 \) then every subgroup \( M \leq \Delta_0 \) is also in \( \Delta_1 \), so every hypermap of type \( \{l_0, m_0, n_0\} \) gives rise to another hypermap of type \( \{l_0, m_0, n_0\} \) of the same genus (since \( M \) is the same group). Moreover, if \( M \leq \Delta_1 \) (so \( M \leq \Delta_0 \)) then we obtain all possible inclusions between pairs of automorphism groups,

\[
\text{Aut } H_0 = \Delta_0 / M \leq \Delta_1 / M = \text{Aut } H_1.
\]

Thus from tables 1, 2, 3 and 4 we deduce all possible inclusions between triangle groups corresponding to genus 2 hypermaps:

\[
\Delta(2, 8, 8) \leq \Delta(2, 4, 8) \leq \Delta(2, 3, 8), \quad \Delta(4, 4, 4) \leq \Delta(2, 4, 8) \leq (2, 3, 8),
\]
\[
\Delta(4, 4, 4) \leq \Delta(3, 3, 4) \leq \Delta(2, 3, 8);
\]
\[
\Delta(3, 3, 6) \leq \Delta(2, 6, 6) \leq \Delta(2, 4, 6), \quad \Delta(4, 4, 3) \leq \Delta(2, 4, 6);
\]
\[
\Delta(5, 5, 5) \leq \Delta(2, 5, 10),
\]
so we get three possible lattices of triangle groups, and hence of automorphism groups of genus 2, namely;

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
These groups have the following relationships:

**LATTICE I:**

(I.1) $C_6 = \langle c \mid c^8 = 1 \rangle$, where $c = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in $GL_2(3)$.

(I.2) $Q_4 = \langle k, l \mid k^4 = 1, k^2 = l^2, klk = l \rangle$, where $k = c^2 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$, $l = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

(I.3) $\langle -2, 4 | 2 \rangle = \langle g, h \mid g^8 = h^2 = 1, hgh = g^3 \rangle$, $g = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $h = c^{-1} l = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$.

(I.4) $GL_2(3) = \langle a, b, c \mid a^2 = b^3 = c^8 = abc = (ac^4)^3 = 1 \rangle$, where $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $c = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$.

(I.5) $SL_2(3) = \langle d, e \mid d^3 = e^4 = 1, ed^2e = ded \rangle$, where $d = (ac^4)^3 c^5 = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$, $e = c^3 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$.

**LATTICE II:**

(II.1) $C_6 = \langle a \mid a^4 = 1 \rangle$. 

Figure 3: Lattices of Automorphism Groups and Triangle Groups.
(II.2). \( C_{10} = \langle b | b^{10} = 1 \rangle \), where \( b^2 = a \).

**LATTICE III:**

(III.1). \( C_8 = \langle a | a^8 = 1 \rangle \).
(III.2). \( C_8 \times C_2 = \langle a, b | a^8 = b^2 = 1, ab = ba \rangle \).
(III.3). \( (4,6,2,2) = \langle a, d | a^6 = d^4 = (da)^2 = (d^{-1}a)^2 = 1 \rangle \).
(III.4). \( \hat{D}_9 = \langle c, d | c^6 = 1, d^2 = c^3, d^{-1}cd = c^5 \rangle \), where \( c = a^3d^2 \) with \( c^2 = b \).

**4. HOMOLOGY REPRESENTATIONS AND CHARACTERS**

We will find the representations \( \rho_i \) and characters \( \tau_i \) on the first integer homology group \( H_1(S, \square) \) of the orientation-preserving automorphism groups \( G \) of regular hypermaps of genus 2. If \( \mathcal{H} \) is a regular hypermap of type \( \{k_o, k_i, k_z\} \) with hypermap subgroup \( N \trianglelefteq \Delta = \Delta(k_o, k_i, k_z) \), then \( G = \text{Aut} \mathcal{H} \cong \Delta/N \) [5]. The natural action of \( G \) on the first integer homology group \( H_1 = H_1(S, \mathbb{Z}) = N/N' \cong \mathbb{Z}^{2g} \) corresponds to the action (induced by conjugation) of \( \Delta/N \) on \( N/N' \). Thus we have a representation of \( G \) on the first homology group, i.e. we simply choose a homology basis on the hypermap and look at the action of \( G \) on this basis (or one can use the Reidemeister-Schreier method [14] to find a presentation for \( N \), then work out the matrix representation of \( G \) on \( H_1(S, \square) \)). On the other hand, the homology character \( \tau_i \) is equal to

\[
\tau_i(x) = \begin{cases} 
2 - \phi(x), & \text{if } x \neq 1, \\
2g & \text{if } x = 1,
\end{cases}
\]

where \( \phi(x) \) is the total number of hypervertices, hyperedges and hyperfaces invariant under \( x \) [16]. Let \( \Delta(k_o, k_i, k_z) \) be the triangle group corresponding to a hypermap of type \( \{k_o, k_i, k_z\} \), then in [18] Macbeath proved that

\[
\phi(x) = |N_G(< x >)| \sum_{i=0}^{2g} \frac{\varepsilon_i(x)}{k_i}
\]

for \( x \neq 1 \), where \( \varepsilon_i(x) = 1 \) or 0 as \( x \) is not conjugate to a power of one of the generators \( x \), of \( G \). Moreover, he also showed that if \( G \) is a cyclic group of order \( n \) and \( x \) has order \( d \), then

\[
\phi(x) = n \sum_{i=1}^{\infty} \frac{k_i^{-1}}{d}
\]
Furthermore, \( \tau_i = d_i \chi_i + \ldots + d_k \chi_k \) for some non-negative integers \( d_i, \ldots, d_k \), where \( \chi_i, \ldots, \chi_k \) are the irreducible complex characters of \( G \) and
\[
d_i = \langle \tau_i, \chi_i \rangle = \frac{1}{|G|} \sum_{x \in G} \tau_i(x) \overline{\chi_i(x)},
\]
the multiplicity of each irreducible character \( \chi_i \) of \( G \) in \( \tau_i \). For more details about the action on the first homology group, we refer to [1], [3], [4], [8], [16], [18], [19], [20].

From the previous chapter we have three lattices of automorphism groups of regular hypermaps of genus 2, so we will consider these lattices.

**LATTICE I:**

(I.1). Let \( G = \langle c \mid c^8 = 1 \rangle \) be the cyclic group of order 8 (the automorphism group of the map \( M_0 \)), where \( c \) is the rotation in the anticlockwise direction by \( 2\pi/8 \) about the central face. If we choose a homology basis \( e_1, e_2, e_3, e_4 \) on the map, see Figure 4, then \( c \) sends \( e_1 \) to \(-e_4\), \( e_2 \) to \( e_1 \), \( e_3 \) to \( e_2 \), and \( e_4 \) to \( e_3 \).

![Figure 4](image1.png)

![Figure 5](image2.png)

Thus we get the following matrix as a representation of \( G \) on the homology group \( H_1(S, \square) \cong \mathbb{Z}^4 \):
\[
\rho_c : c \mapsto \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix} = C \in GL_4(\mathbb{Z}).
\]

If we apply the formula (2) to \( C, \), we see that \( \phi(x) = 2, 2, 6 \) as \( x \in C_8 \) has order 8, 4, 2 (for example, the vertex and face of \( M_0 \) are invariant under the elements
\(x = c, c', c^3, c^5\) and \(c^7\) of order 8, so \(\phi(x) = 2\). Then (1) gives the character-values
\(\tau_i(x) = 4, 0, 0, -4\) as \(x \in C_s\) has order 1, 8, 4, 2 (as we can see from the trace \(\text{tr} (\rho_i)\) of \(\rho_i\)). Then we get \(\tau_i = \chi_1 \pm \chi_3 \pm \chi_5 \pm \chi_7\) with values in the cyclotomic field \(Q(e^{\pi i/4})\), where \(\chi_1, \chi_3, \chi_5, \chi_7\) are irreducible (faithful) complex characters of \(G\) (i.e. \(\chi_i(c) = w \cdot w = e^{\pi i/4}\)).

On the other hand, if we choose a reflection line \(\ell\) on the map, see Figure 4, we see that the matrix \(\alpha_i\) of the orientation-reversing automorphism \(\alpha\) with respect to the chosen homology basis is

\[
\alpha_i = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

(I.2). Let \(G = Q_s = \langle k, l | k^4 = 1, k^2 l^2, klk = l \rangle\) be the quaternion group of order 8 (the automorphism group of the hypermap \(H_3\), where \(k\) and \(l\) are the rotations by \(\pi/2\) about the central face and the other face, respectively. Hence the six elements of order 4 in \(Q_s\) are the quarter-turns fixing the centres of the two hypervertices, hyperedges, and hyperfaces, respectively, so the element \(k^2\) rotates each of these through a half-turn.

We choose a homology basis \(\{e_1, e_2, e_3, e_4\}\) on \(H_3\) (see Figure 5), then \(k\) sends \(e_1 \mapsto -e_4, e_2 \mapsto -e_3, e_3 \mapsto e_1, e_4 \mapsto e_2\); and \(l\) sends \(e_1 \mapsto -e_2 - e_3 - e_4, e_2 \mapsto e_1 + e_2 + e_3, e_3 \mapsto -e_1 - e_2 + e_4, e_4 \mapsto e_1 - e_2 - e_4\).

Thus we obtain the following homology representation of \(G\)

\[
\rho_i : k \mapsto \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} = K; \quad l \mapsto \begin{pmatrix}
0 & -1 & -1 & -1 \\
1 & 1 & 1 & 0 \\
-1 & -1 & 0 & 1 \\
1 & 0 & -1 & -1
\end{pmatrix} = L.
\]

Furthermore, it is not difficult to see from Figure 5 that \(\phi(x) = 2, 6\) as \(x \in Q_s\) has order 4, 2, respectively. Then (1) gives \(\tau_i(x) = 4, 0, -4\) as \(x\) has order 1, 4, 2, (also from the representation \(\rho_i\)), respectively. Thus we have \(\tau_i = 2\chi_5\), where \(\chi_5\) is an irreducible complex character of \(Q_s\) given by \(\chi_5(k) = 2, \chi_5(k^3) = -2, \chi_5(k^5) = \chi_5(l) = \chi_5(kl) = 0\). Here \(1, k^2, k, l, kl\) are the representatives of conjugacy classes of \(G\).

If we choose a reflection line \(\ell\) on the hypermap (see Figure 5) then we get the same matrix \(\alpha_i\) with respect to chosen homology basis as in (I.1).
(1.3). Let $G = \langle -2, 4 \mid 2 >, h \mid g^8 = h^3 = 1, hgh = g^3 >$, of order 16 (the orientation-preserving automorphism group of $M_3$), where $g$ is a rotation through $2\pi/8$ about the central face and $h$ is a rotation about the midpoint of an edge, see Figure 6.

As usual, we choose a homology basis on this map, then we find the action of the automorphism group on these basis elements. It follows immediately from Figure 6 that $g$ sends $e_i \mapsto -e_i$, $e_2 \mapsto e_1$, $e_3 \mapsto e_2$, and $e_4 \mapsto e_3$, and similarly $h$ sends $e_i \mapsto -e_i + e_3 - e_4$, $e_1 \mapsto e_1 - e_2 + e_3 - e_4$, $e_2 \mapsto e_1 - e_2 + e_3$, $e_3 \mapsto e_1 - e_2 + e_3$, and $e_4 \mapsto e_1 - e_2 + e_4$.

![Figure 6](image)

Then the homology representation $\rho_1$ of $G$ follows:

$$
\rho_1: g \mapsto \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix} = G : h \mapsto \begin{pmatrix}
-1 & 0 & 1 & -1 \\
0 & -1 & 1 & -1 \\
1 & -1 & 1 & 0 \\
1 & -1 & 0 & 1
\end{pmatrix} = H \in \text{GL}_4(\mathbb{Z}).
$$

Moreover, the homology character is equal to $\tau_1 = \chi_6 + \chi_7$, where $\chi_6$ and $\chi_7$ are irreducible complex characters of $G$ given by $\chi_6(1) = 2$, $\chi_6(g^4) = -2$, $\chi_6(g^2) = \sqrt{-2}$, $\chi_6(g^3) = -\sqrt{-2}$, $\chi_6(h) = 0$, $\chi_6(gh) = 0$, and $\chi_7(1) = 2$, $\chi_7(g^4) = -2$, $\chi_7(g^2) = 0$, $\chi_7(g) = -\sqrt{-2}$, $\chi_7(g^3) = \sqrt{-2}$, $\chi_7(h) = \chi_7(gh) = 0$, where $1, g^4, g^2, g, gh$ are the representatives of conjugacy classes of $G$.

On the other hand, if we choose a reflection line $l$ on the map, see Figure 6, we find that the matrix $\alpha_1$ corresponding to this reflection line and the homology basis is
\[ \alpha_i = \begin{pmatrix}
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 
\end{pmatrix}. \]

(I.4) Let \( G = \text{GL}_2(3) = \langle a, b, c \mid a^2 = b^3 = c^4 = abc = (ac^4)^2 = 1 \rangle \) of order 48, the automorphism group of \( \mathbb{M}_5 \).

From Figure 7, we get the homology representation and character of \( G \) as follows:

\[ \rho_i : a \mapsto \begin{pmatrix}
-1 & -1 & 0 & 1 \\
0 & 0 & -1 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & -1 & 0 
\end{pmatrix} = A ; \quad b \mapsto \begin{pmatrix}
-1 & -1 & -1 & 0 \\
1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & -1 
\end{pmatrix} = B ; \]

\[ c \mapsto \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 
\end{pmatrix} = \text{C} \in \text{GL}_4(\mathbb{F}) , \]

and \( \tau_i = \chi_7 + \chi_8 \), where \( \chi_7 \) and \( \chi_8 \) are irreducible complex characters of \( G \) given by \( \chi_7(1) = 2, \chi_7(c^4) = -2, \chi_7(c^2) = 0, \chi_7(c) = \sqrt{-2}, \chi_7(c^7) = -\sqrt{-2}, \chi_7(a) = 0, \chi_7(b) = -1, \chi_7(c^4b) = 1, \) and \( \chi_8(1) = 2, \chi_8(c^4) = -2, \chi_8(c^2) = 0, \chi_8(c) = -\sqrt{-2}, \chi_8(c^7) = \sqrt{-2}, \chi_8(a) = 0, \chi_8(b) = -1, \chi_8(c^4b) = -1 \) (where \( 1, c^4, c^2, c^7, a, b, c^4b \) are the representatives of conjugacy classes of \( G \)).

Moreover, if we choose a reflection line \( \ell \) on the map (see Figure 7), then it is not difficult to see that the matrix \( \alpha_i \) of the orientation-reversing automorphism \( \alpha \) is the same as in (I.1).

(I.5) Let \( G = \text{SL}_2(3) = \langle d, e \mid d^3 = e^4 = 1, ed^3e = d ed \rangle \) be the special linear group of order 24, the automorphism group of \( \mathbb{H}_5 \), where \( d = (ac^5)c^5 \) and \( e = c^3 \).

Then it is easily verified from Figure 8 that the homology representation of \( G \) is

\[ \rho_i : d \mapsto \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & -1 & 0 & 0 \\
1 & 0 & -1 & -1 \\
0 & 1 & 1 & 0 
\end{pmatrix} = D ; \quad e \mapsto \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 
\end{pmatrix} = E , \]
and then $\tau_1 = 2 \chi_5$, where $\chi_5$ is an irreducible complex character of $G$ given by $\chi_5(1) = 2$, $\chi_5(e^2) = -2$, $\chi_5(e) = 0$, $\chi_5(d) = -1$, $\chi_5(d^2) = -1$, $\chi_5(de^2) = 1$, $\chi_5(d^2e^2) = 1$, (where $1, e^2, e, d, d^2, d^2e^2, de^2$ are the representatives of conjugacy classes of $G$).

It can be seen that the matrix $\alpha_1$ of the orientation-reversing automorphism $\alpha$ corresponding to a chosen reflection line $\ell$ (see Figure 8) on the hypermap with respect to the homology basis is the same as in (1.3).

\begin{align*}
\text{Figure 8} & \quad \text{Figure 9}
\end{align*}

**LATTICE II:**

(II.1). Let $G = \langle a \mid a^5 = 1 \rangle$ be the cyclic group of order 5, the automorphism group of the hypermap $H_1$, where $a$ is a rotation through $2\pi/5$ about the centre.

Let $e_1 = x_1 + x_2$, $e_2 = -x_1 + x_5$, $e_3 = -x_4 - x_3$ and $e_4 = x_3 + x_4$ be a chosen homology basis on the hypermap, see Figure 9. Then we obtain the homology representation as

$$
\rho_1 : a \mapsto \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & -1 & -1 & -1 \\
1 & 0 & 0 & 0
\end{pmatrix} = A \in \text{GL}_4(\mathbb{C})
$$

and then the homology character as $\tau_1 = \chi_1 + \chi_2 + \chi_3 + \chi_4$, where $\chi_1, \chi_2, \chi_3$ and $\chi_4$ are irreducible (faithful) complex characters of $G$. 
Furthermore, if we choose a reflection line \( l \) on the hypermap, see Figure 9, then we see that the matrix \( \alpha \) of the orientation-reversing automorphism \( \alpha \) with respect to the homology basis is

\[
\alpha = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(II.2). Let \( G = \langle b \mid b^{10} = 1 \rangle \) be the cyclic group of order 10, the automorphism group of the map \( M_1 \), where \( b \) is a rotation through \( 2\pi / 10 \) about the centre.

Let us choose a homology basis on the map, see Figure 10, such as \( e_1 = x_1 + x_3 \), \( e_2 = -x_1 + x_3 \), \( e_3 = -x_4 - x_5 \) and \( e_4 = x_3 + x_4 \). Then it is not difficult to see that the homology representation of \( G \) is

\[
\rho : b \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 1 & 1 & 1
\end{pmatrix} = B \in \text{GL}_4(\mathbb{C})
\]

and \( \tau = \chi_1 + \chi_3 + \chi_7 + \chi_9 \), where \( \chi_1, \chi_3, \chi_7 \) and \( \chi_9 \) are irreducible (all faithful) complex characters of \( G \).

The matrix \( \alpha \) of the orientation-reversing automorphism \( \alpha \) corresponding to the chosen reflection line \( l \) with respect to the homology basis is the same as in (II.1).

**LATTICE III:**

(III.1). Let \( G = C_6 = \langle a \mid a^6 = 1 \rangle \) be the cyclic group of order 6, the automorphism group of the hypermap \( H_2 \), where \( a \) is a rotation through \( 2\pi / 6 \) about the central hyperface, see Figure 11.

Let us choose a homology basis on the hypermap such as \( e_1 = x_1 + x_5 \), \( e_2 = x_1 - x_4 \), \( e_3 = -x_2 - x_5 \), \( e_4 = -x_2 - x_3 \).
Then it follows that

\[ \rho_1 : a \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = A \in GL_4(\mathbb{C}), \]

and

\[ \tau_1 = \chi_1 + \chi_2 + \chi_4 + \chi_5, \]

where \( \chi_1, \chi_2, \chi_4 \) and \( \chi_5 \) are irreducible complex characters of \( G \).

Moreover, the matrix \( \alpha_1 \) of the orientation-reversing automorphism \( \alpha \) with respect to the chosen reflection line \( t \) and the homology basis is

\[ \alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]

**III.2.** Let \( G = \mathbb{Z}_6 \times \mathbb{Z}_2 = \langle a, b \mid a^6 = b^2 = 1, ab = ba \rangle \) be the abelian group of order 12, the automorphism group of the map \( M_2 \), where \( a \) is a rotation of Figure 12 by \( 2\pi/6 \) and \( b \) is a half-turn reversing each edge and transposing the two faces and the vertices.

Let us choose a homology basis on the map as \( e_1 = x_1 - x_2, e_2 = x_5 - x_2, e_3 = x_6 - x_5, e_4 = x_4 - x_1 \), see Figure 12.
Then we get the following homology representation for $G$:

$$
\rho_i : a \mapsto \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} = A; \quad b \mapsto \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} = B,
$$

and then $\tau_i = \chi_{1,1} + \chi_{2,1} + \chi_{4,1} + \chi_{5,1}$, where $\chi_{1,1}, \chi_{2,1}, \chi_{4,1}$ and $\chi_{5,1}$ are irreducible complex characters of $G$ given by

\[
\begin{align*}
\chi_{1,1} & : 1 \quad w \quad w^2 \quad -1 \quad w^4 \quad w^5 \\
\chi_{2,1} & : 1 \quad w^2 \quad w^4 \quad 1 \quad w^2 \quad -1 \quad -w^2 \quad -w^4 \\
\chi_{4,1} & : 1 \quad w^4 \quad w^2 \quad 1 \quad w^4 \quad w^2 \quad -1 \quad -w^4 \quad -w^2 \\
\chi_{5,1} & : 1 \quad w^5 \quad w^4 \quad -1 \quad w^3 \quad w \quad -1 \quad -w^3 \quad -w
\end{align*}
\]

where $w = e^{2\pi i} = \frac{1 + \sqrt{-3}}{2}$.

If we choose a reflection line $\ell$, see Figure 12, it can be seen that the corresponding matrix $\alpha_i$ is the same as in (III.1).

(III.3) Let $G = \left\langle 4, 6 \mid 2, 2 \right\rangle = < a, d \mid a^6 = d^4 = (da)^2 = (d^{-1} a)^2 = 1 >$ of order 24, the automorphism group of the map $M_4$. 

We choose a homology basis on the map as \( e_1 = x_1 - x_2 - x_3 - x_6, \)
\( e_2 = x_1 - x_2 - x_3 - x_4, \)
\( e_3 = x_6 - x_2 - x_4 - x_5 \) and \( e_4 = x_6 - x_1 - x_2 - x_3, \) see Figure 13.

Then the matrix representation \( \rho_i \) of \( G \) with respect to the chosen basis is

\[
\rho_i : a \mapsto \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} = A; \quad d \mapsto \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} = D,
\]

and \( \tau_i = \chi_6 + \chi_7, \) where \( \chi_6 \) and \( \chi_7 \) are irreducible complex characters of \( G \) given by

\[
\chi_6(1) = 2, \quad \chi_6(b^2) = -2, \quad \chi_6(a^2) = \sqrt{-3}, \quad \chi_6(a^3) = 0, \quad \chi_6(a^4) = \sqrt{-3},
\]

\[
\chi_7(a^2) = -1, \quad \chi_7(a^3) = 1, \quad \chi_7(b) = 0, \quad \chi_7(ab) = 0, \quad \text{and} \quad \chi_7(1) = 2, \quad \chi_7(b^2) = -2,
\]

\[
\chi_7(a) = -\sqrt{-3}, \quad \chi_7(a^2) = 0, \quad \chi_7(a^3) = \sqrt{-3}, \quad \chi_7(a^4) = -1, \quad \chi_7(a^5) = 1,
\]

\[
\chi_7(b) = 0, \quad \chi_7(ab) = 0, \quad \text{where} \quad 1, d^2, a, a^3, a^4, a^5, d, ad \quad \text{are the representatives}
\]

of conjugacy classes of \( G \).

If we choose a reflection line \( l \) on the map, see Figure 13, we see that the matrix \( \alpha_i \) of the orientation-reversing automorphism \( \alpha \) is the same as in case (III.1).

(III.4). Finally, let \( G = \tilde{D}_2 = \langle c, d \mid c^6 = 1, d^2 = c^3, d^{-1}cd = c^5 \rangle \) be the binary dihedral group of order 12, the automorphism group of the hypermap \( H_4 \).
Since \( D_3 \leq \langle 4, 6 \rangle \) with \( c = a^2b^2 \) and \( c^2 = b \), then the homology representation follows:
\[
\rho_1 : c \mapsto \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = C; \quad d \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = D,
\]
and \( \tau_1 = 2\chi_s \) where \( \chi_s \) is an irreducible complex character of \( G \) given by,
\( \chi_s(1) = 2, \chi_s(c') = 2, \chi_s(c) = -1, \chi_s(c^2) = -1, \chi_s(d) = 0, \chi_s(cd) = 0 \), where
\( 1, c', c, c^2, d, cd \) are the representatives of conjugacy classes of \( G \).

Let us choose a homology basis such as \( e_i = x_i - x_2 - x_3 - x_4 \),
\( e_2 = x_1 - x_2 - x_3 - x_4 \), \( e_3 = x_3 - x_2 - x_4 - x_1 \), \( e_4 = x_4 - x_1 - x_3 - x_2 \), and a reflection line \( \ell \) on the hypermap, see Figure 14. Then the matrix \( \alpha_1 \) corresponding to this orientation-reversing automorphism \( \alpha \) is the same as in case (III.1).

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