ON THE GEOMETRY OF SPACELIKE CONGRUENCES

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ABSTRACT

The Blaschke vectors of the Blaschke trihedrons of spacelike and timelike ruled surfaces defined in [10]–[12].
In this study we give some fundamental formulae and facts about the geometry of spacelike congruences by means of dual hyperbolic and central angles.

1. INTRODUCTION

Dual numbers had been introduced by W. K. Clifford as a tool for his geometrical investigations. After him E. Study used dual numbers and dual vectors in this research on the geometry of lines and kinematics. He devoted special attention to the representation of directed lines by dual unit vectors and defined the mapping which is called with his name. The set of directed lines in the Euclidean 3-space \( R^3 \) is one to one correspondence with the points of dual unit sphere at dual space \( D^3 \) of triples of dual numbers (E. Study’s mapping) [6]. A differentiable curve on dual unit sphere \( S^2 \) of \( D^3 \) corresponds to a ruled surface in space of lines \( R^3 \) [13]. A dual unit vector shows a congruence in \( R^3 \) if it is connected to parameters \( u \) and \( v \). A. Çalışkan gave the relation between Blaschke vectors of the parameter ruled surfaces and an arbitrary ruled surface passing through a generator of a congruence of lines at \( D^3 \), and some corollaries [4].

If we take the space of Lorentzian lines \( R^3_1 \) with signature \((+,-,-)\) instead of the space of lines \( R^3 \) we can state the E. Study’s mapping as follows: The dual timelike and spacelike unit vectors of dual hyperbolic and Lorentzian unit spheres \( H^2_0 \) and \( S^2 \) at dual Lorentzian space \( D^3_1 \) are one to one correspondence with the directed timelike and spacelike lines of the space of Lorentzian lines, respectively [7]. Then a differentiable curve on \( H^2_0 \) corresponds to a timelike ruled surface while a differentiable timelike (resp. spacelike) curve on \( S^2 \) corresponds to a space like (resp. timelike) ruled surface at \( R^3_1 \).
In present paper we consider the Blaschke vectors of Blaschke trihedrons of time like and spacelike ruled surfaces at their striction points, similar to the Darboux vectors defined for spacelike and timelike curves on a timelike surface at $R^3_1$ [8]. Using these vectors we obtain two fundamental formulae between Blaschke vectors of arbitrary spacelike(resp. timelike) ruled surface and parameter ruled surfaces passing through a generator of a spacelike congruence $\tilde{a}(u,v)$. From this formulae, Manheim’s and Liouville’s formulae, and some corollaries are obtained.

The invariants on a spacelike congruence will be given in other paper [9].

1. Preliminaries

We start with preliminaries on the geometry of Minkowski 3-space. Let $R^3_1$ be a Minkowski 3-space endowed with Lorentzian inner product $\langle \cdot, \cdot \rangle$ of signature $(+,+,-)$. A vector $a = (a_1, a_2, a_3)$ of $R^3_1$ is said to be timelike if $\langle a, a \rangle < 0$, space-like if $\langle a, a \rangle > 0$ and lightlike (or null) if $\langle a, a \rangle = 0$. The set of all vectors $a$ such that $\langle a, a \rangle = 0$ is called the lightlike (or null) cone and is denoted by $\wedge$. The norm of a vector $a$ is defined to be $|a| = \sqrt{\langle a, a \rangle}$. Time orientation is defined as follows: A timelike vector $a = (a_1, a_2, a_3)$ is future pointing (resp. past pointing) if and only if $a_3 > 0$ (resp. $a_3 < 0$) [2]. The hyperbolic and Lorentzian spheres of radius 1 in $R^3_1$ are given by

$$H^2_0 = \left\{ a = (a_1, a_2, a_3) \in R^3_1 / \langle a, a \rangle = -1 \right\}$$

and

$$S^2_1 = \left\{ a = (a_1, a_2, a_3) \in R^3_1 / \langle a, a \rangle = 1 \right\},$$

respectively [5].

**Lemma 1.** Let $a$ and $b$ be two future pointing (resp. past pointing) timelike unit vectors in $R^3_1$. Then we have

$$\langle a, b \rangle = -\cosh \theta$$

([2], [5]).

As in the case of Euclidean 3-space, the Lorentzian cross product of $a$ and $b$ is defined by

$$a \wedge b = (b_2 a_3 - a_2 b_3, a_1 b_3 - a_3 b_1, a_1 b_2 - a_2 b_1),$$

(1.1)

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are the vectors of the space $R^3_1$ [1]. This product satisfies the equalities
\[ \langle a \wedge b, c \rangle = \det(a, b, c), \]
\[ (a \wedge b) \wedge c = -\langle a, c \rangle b + \langle b, c \rangle a \]
and
\[ \langle a \wedge b, c \wedge d \rangle = -\langle a, c \rangle \langle b, d \rangle + \langle a, d \rangle \langle b, c \rangle. \]

We know that a dual number has the form \( a + \varepsilon a^* \), where \( a \) and \( a^* \) are real numbers and \( \varepsilon \) is the dual unit with the property \( \varepsilon^2 = 0 \). The set of all dual numbers is a ring denoted by \( D \). The set of triples of dual numbers
\[ D^3 = \{ \tilde{a} = (a_1, a_2, a_3) / a_i \in D, 1 \leq i \leq 3 \} \]
is a module over the ring \( D \) which is called \( D \)-module or dual space. The elements of \( D^3 \) are called as dual vectors. A dual vector \( \tilde{a} \) may be written in the form \( \tilde{a} = a + \varepsilon a^* \), where \( a \) and \( a^* \) are the vectors of \( R^3 \).

The Lorentzian inner product of two dual vector \( \tilde{a} = a + \varepsilon a^* \) and \( \tilde{b} = b + \varepsilon b^* \) is defined as
\[ \langle \tilde{a}, \tilde{b} \rangle = \langle a, b \rangle + \varepsilon (\langle a, b^* \rangle + \langle a^*, b \rangle), \]
where \( \langle a, b \rangle \) is the Lorentzian inner product of vectors \( a \) and \( b \) in Minkowski 3-space \( R^3 \) with signature \((+,-,-)\) [7]. A dual vector \( \tilde{a} \) is said to to be timelike if \( \langle \tilde{a}, \tilde{a} \rangle > 0 \), spacelike if \( \langle \tilde{a}, \tilde{a} \rangle > 0 \) and lightlike (or null) if \( \langle \tilde{a}, \tilde{a} \rangle = 0 \). The set of all vectors \( \tilde{a} \) such that \( \langle a, a \rangle = 0 \) is called the dual lightlike (or dual null) cone and is denoted by \( \mathcal{V} \). The norm of a dual vector \( \tilde{a} \) is defined to be
\[ |\tilde{a}| = |a| + \varepsilon \frac{\langle a, a^* \rangle}{|a|} \]
[7].

We also consider the time orientation as follows: A dual timelike vector \( \tilde{a} \) is future pointing (resp. past pointing) if and only if \( a \) is future pointing (resp. past pointing). We call dual Lorentzian space the set of all dual timelike, spacelike and lightlike vectors and denote it by \( D^3_1 \). Then we define the dual hyperbolic and Lorentzian spheres of radius 1 in \( D^3_1 \) by
\[ H^2_0 = \{ \tilde{a} = a + \varepsilon a^* \in D^3_1 / \langle \tilde{a}, \tilde{a} \rangle = -1 \} \]
and
\[ S^2_1 = \{ \tilde{a} = a + \varepsilon a^* \in D^3_1 / \langle \tilde{a}, \tilde{a} \rangle = 1 \} \]
respectively [7]. There are two component of sphere $H_0^2$. We call the components of $H_0^2$ passing through $(0,0,1)$ and $(0,0,-1)$ as future pointing dual hyperbolic unit sphere and past pointing dual hyperbolic unit sphere and denote by $H_0^{2+}$ and $H_0^{2-}$, respectively. With respect to this definition, we can write:

$$H_0^{2+} = \{ \tilde{a} = a + \varepsilon a^* \in H_0^2 / a \text{ is a future pointing vector} \},$$

$$H_0^{2-} = \{ \tilde{a} = a + \varepsilon a^* \in H_0^2 / a \text{ is a past pointing vector} \}.$$

As in the case of the space $R_i^3$ we define Lorentzian cross product of dual vectors $\tilde{a}$ and $\tilde{b}$ by

$$\tilde{a} \wedge \tilde{b} = a \wedge b + \varepsilon (a \wedge b^* + b^* \wedge a),$$

where $a \wedge b$ is the Lorentzian cross product given by (1.1). Then we give the correspondences of equalities (1.2) - (1.4):

$$<\tilde{a} \wedge \tilde{b}, \tilde{c}> = \det(\tilde{a}, \tilde{b}, \tilde{c}),$$

$$<\tilde{a} \wedge \tilde{b} \wedge \tilde{c} > = -<\tilde{a}, \tilde{c}> \tilde{b} + <\tilde{b}, \tilde{c}> \tilde{a},$$

$$<\tilde{a} \wedge \tilde{b}, \tilde{c} \wedge \tilde{d} > = -<\tilde{a}, \tilde{c}> <\tilde{b}, \tilde{d} > + <\tilde{a}, \tilde{d}> <\tilde{b}, \tilde{c}>$$

[7].

2. Dual Lorentzian Spherical Curves, Timelike and Spacelike Ruled Surfaces

A timelike ruled surface is defined as a surface generated by the motion of straight timelike line while a spacelike ruled surface is defined as a surface generated by the motion of a straight spacelike line in $R_i^3$. Using E. Study's mapping for the elements of Lorentzian spaces $D_i^3$ and $R_i^4$, spacelike (resp. timelike) ruled surfaces are represented by a dual spacelike (resp. timelike) unit vector of an arbitrary real parameter $t$:

$$\tilde{a}(t) = a(t) + \varepsilon a^*(t).$$

This means that the image of a spacelike (resp. timelike) ruled surface is a timelike (resp. spacelike) curve on dual Lorentzian unit sphere $S_1^2$. Therefore the terms dual Lorentzian spherical curve, timelike and spacelike ruled surfaces are synonymous in this paper.

W. Blaschke defined in [3] the trihedron of a ruled surface at striction point which is called with his name. In Blaschke trihedron $[\tilde{a}_1, \tilde{a}_2, \tilde{a}_3]$, the vectors $\tilde{a}_i$ ($i=1,2,3$) is the generator, the normal and the tangent passing through striction point P of the ruled surface, respectively.
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i) We suppose that in Blaschke trihedron \( [\tilde{a}_1, \tilde{a}_2, \tilde{a}_3] \), \( \tilde{a}_2 \) is timelike unit vector. Then we can write
\[
\begin{align*}
<\tilde{a}_i, \tilde{a}_i> &= 1, \quad (i = 1, 3); \quad <\tilde{a}_2, \tilde{a}_2> = -1 \\
<\tilde{a}_i, \tilde{a}_j> &= 0, \quad (i \neq j)
\end{align*}
\]  
(2.1)

and
\[
\tilde{a}_1 \wedge \tilde{a}_2 = -\tilde{a}_3, \quad \tilde{a}_2 \wedge \tilde{a}_3 = -\tilde{a}_1, \quad \tilde{a}_3 \wedge \tilde{a}_1 = -\tilde{a}_2.
\]  
(2.2)

The Blaschke derivative formulae are given by
\[
\begin{align*}
\tilde{a}_1' &= \tilde{p} \tilde{a}_2, \\
\tilde{a}_2' &= \tilde{p} \tilde{a}_1 + \tilde{q} \tilde{a}_3, \\
\tilde{a}_3' &= \tilde{q} \tilde{a}_2 \\
\tilde{p} &= \sqrt{<\tilde{a}_1', \tilde{a}_1'>} \\
\tilde{q} &= \frac{(\tilde{a}_1, \tilde{a}_1', \tilde{a}_3')}{<\tilde{a}_1', \tilde{a}_1'>}
\end{align*}
\]  
(2.3)

where \( \tilde{p} \) and \( \tilde{q} \) are dual curvature and dual torsion, respectively.

The matrix form of formulae (2.3) is
\[
\begin{bmatrix}
\tilde{a}_1' \\
\tilde{a}_2' \\
\tilde{a}_3'
\end{bmatrix}
= 
\begin{bmatrix}
0 & \tilde{p} & 0 \\
\tilde{p} & 0 & \tilde{q} \\
0 & \tilde{q} & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_3
\end{bmatrix}
\]  
(2.4)

By (2.4) we write the Blaschke vector of the Blaschke trihedron
\[
\tilde{b} = -\tilde{q} \tilde{a}_1 + \tilde{p} \tilde{a}_3
\]  
(2.5)

given in [12]. In this case, the formulae (2.3) can be written by the vector (2.5) as follows:
\[
\tilde{a}_i' = \tilde{b} \wedge \tilde{a}_i.
\]  
(2.6)

ii) In Blaschke trihedron \( [\tilde{a}_1, \tilde{a}_2, \tilde{a}_3] \), let \( \tilde{a}_3 \) be timelike unit vector. Then we can write
\[
\begin{align*}
<\tilde{a}_i, \tilde{a}_i> &= 1, \quad (i = 1, 2); \quad <\tilde{a}_3, \tilde{a}_3> = -1 \\
<\tilde{a}_i, \tilde{a}_j> &= 0, \quad (i \neq j)
\end{align*}
\]  
(2.7)

and
\[
\tilde{a}_1 \wedge \tilde{a}_2 = \tilde{a}_3, \quad \tilde{a}_2 \wedge \tilde{a}_3 = -\tilde{a}_1, \quad \tilde{a}_3 \wedge \tilde{a}_1 = -\tilde{a}_2.
\]  
(2.8)

The Blaschke derivative formulae are given by
\[
\begin{align*}
\tilde{a}_1' &= \tilde{p} \tilde{a}_2, \\
\tilde{a}_2' &= -\tilde{p} \tilde{a}_1 + \tilde{q} \tilde{a}_3, \\
\tilde{a}_3' &= \tilde{q} \tilde{a}_2 \\
p &= \sqrt{<\tilde{a}_1', \tilde{a}_1'>} \\
q &= \frac{(\tilde{a}_1, \tilde{a}_1', \tilde{a}_3')}{<\tilde{a}_1', \tilde{a}_1'>}
\end{align*}
\]  
(2.9)

The matrix form of the formulae (2.9) is
\[
\begin{bmatrix}
\tilde{a}_1' \\
\tilde{a}_2' \\
\tilde{a}_3'
\end{bmatrix} =
\begin{bmatrix}
0 & \tilde{p} & 0 \\
-\tilde{p} & 0 & \tilde{q} \\
0 & \tilde{q} & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_3
\end{bmatrix}
\]  
(2.10)

By (2.10) we write the Blaschke vector
\[
\tilde{b} = \tilde{q} \tilde{a}_1 - \tilde{p} \tilde{a}_3
\]  
(2.11)
given in [10]. In this case, the formulae (2.9) can be written by the vector (2.11) as follows:
\[
\tilde{a}_i' = \tilde{b} \wedge \tilde{a}_i.
\]  
(2.12)

3. Spacelike Congruences

If a dual spacelike unit vector \( \tilde{a} = a + \epsilon a^* \) of the sphere \( S^2_1 \) is connected to the parameter \( u \) and \( v \) then \( \tilde{a}(u,v) = a(u,v) + \epsilon a^*(u,v) \) shows a spacelike congruence which its generators are dual spacelike unit vectors at dual Lorentzian space \( D^3_1 \). Let us denote an arbitrary spacelike ruled surface, spacelike parameter ruled surface \( v = \) constant and timelike parameter ruled surface \( u = \) constant passing through a straight line of the congruence \( \tilde{a}(u,v) \) by \( (\tilde{c}_1) \), \( (\tilde{c}_2) \) and \( (\tilde{c}_3) \), respectively. We suppose that parameter ruled surfaces are orthogonal. Let Blaschke trihedrons of these ruled surfaces be \([\tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{13}]\), \([\tilde{a}_{21}, \tilde{a}_{22}, \tilde{a}_{23}]\) and \([\tilde{a}_{31}, \tilde{a}_{32}, \tilde{a}_{33}]\), respectively, where \( \tilde{a}_0 = \tilde{a}_{11} = \tilde{a}_{21} = \tilde{a}_{31} \). The second components of this trihedrons are the normals of ruled surfaces while third components are the tangents of surface as in [12]. Then we can write
\[
\tilde{a}_0 \wedge \tilde{a}_{12} = -\tilde{a}_{13}, \quad \tilde{a}_0 \wedge \tilde{a}_{22} = -\tilde{a}_{23}, \quad \tilde{a}_0 \wedge \tilde{a}_{32} = \tilde{a}_{33}.
\]  
(3.1)

By (2.5) and (2.11) the Blaschke vectors of these ruled surfaces are given by
\[
\tilde{b}_i = -\tilde{q}_i \tilde{a}_{i1} + \tilde{p}_i \tilde{a}_{i3} \quad \text{;} \quad i=1,2
\]
and
\[
\tilde{b}_3 = \tilde{q}_3 \tilde{a}_{13} - \tilde{p}_3 \tilde{a}_{33}
\]  
(3.2)
respectively.

Let \( \tilde{s}_i \) \((i=1,2,3)\) be the dual arc lengths of ruled surfaces \((\tilde{c}_i)\). Thus it can be written
\[
\hat{a}_{12} = \hat{a}_u \frac{du}{d\hat{s}_1} + \hat{a}_v \frac{dv}{d\hat{s}_1} \\
\hat{a}_{22} = \frac{\hat{a}_u}{\lVert \hat{a}_u \rVert} = \frac{\hat{a}_u}{\sqrt{\hat{e}}} \\
\hat{a}_{32} = \frac{\hat{a}_v}{\lVert \hat{a}_v \rVert} = \frac{\hat{a}_v}{\sqrt{\hat{g}}},
\]

and

\[
\hat{a}_{22} \land \hat{a}_{32} = \frac{\hat{a}_u \land \hat{a}_v}{\lVert \hat{a}_u \land \hat{a}_v \rVert} = \hat{a}_0.
\]

(3.4)

By the last two equation of (3.3) we have

\[
\begin{align*}
\hat{a}_u &= \sqrt{\hat{e}} \hat{a}_{22} \\
\hat{a}_v &= \sqrt{\hat{g}} \hat{a}_{32}.
\end{align*}
\]

(3.5)

Writing the equalities (3.5) in the first of (3.3) we obtain

\[
\hat{a}_{12} = \sqrt{\hat{e}} \hat{a}_{22} \frac{du}{d\hat{s}_1} + \sqrt{\hat{g}} \hat{a}_{32} \frac{dv}{d\hat{s}_1}.
\]

(3.6)

Let's denote the \textit{dual hyperbolic angle} between dual timelike vectors \(\hat{a}_{12}\) and \(\hat{a}_{22}\) by \(\tilde{\vartheta}\).

Then we can write

\[
\hat{a}_{12} = \hat{a}_{22} \cosh \tilde{\vartheta} + \hat{a}_{32} \sinh \tilde{\vartheta}.
\]

(3.7)

Multiplying the equation (3.7) by \(\hat{a}_{22}\) and \(\hat{a}_{32}\) we find

\[
\begin{align*}
<\hat{a}_{12}, \hat{a}_{22}> &= - \cosh \tilde{\vartheta} = - \sqrt{\hat{e}} \frac{du}{d\hat{s}_1} \\
<\hat{a}_{12}, \hat{a}_{32}> &= \sinh \tilde{\vartheta} = \sqrt{\hat{g}} \frac{dv}{d\hat{s}_1}.
\end{align*}
\]

(3.8)

Furthermore, for the arcs \(d\hat{s}_i\) we can write

\[
\begin{align*}
d\hat{s}_1^2 &= \tilde{\varepsilon} du^2 - \tilde{\varrho} dv^2, \quad \tilde{f} = 0 \\
d\hat{s}_2^2 &= \tilde{\varepsilon} du^2, \quad \nu = \text{cons.} \quad dv = 0 \\
d\hat{s}_3^2 &= \tilde{\varrho} dv^2, \quad \upsilon = \text{cons.} \quad du = 0,
\end{align*}
\]

(3.9)

where \(\tilde{\varepsilon} = e + \varepsilon e^*\), \(\tilde{f} = f + \varepsilon f^*\) and \(\tilde{\varrho} = g + \varepsilon g^*\) are the coefficients of the Lorentzian inner product on \(\hat{a}(u, v)\).
By the formulae (3.8) and (3.9) it is seen that
\[
\begin{align*}
\frac{\partial \tilde{s}_2}{\partial \tilde{s}_1} &= \cosh \tilde{\theta} \\
\frac{\partial \tilde{s}_3}{\partial \tilde{s}_1} &= \sinh \tilde{\theta}.
\end{align*}
\] (3.10)

Multiplying the equation (3.7) by \( \tilde{a}_0 \) and considering that \( \tilde{a}_{i3} = \tilde{a}_{i2} \wedge \tilde{a}_0 \) we find the equality
\[
\tilde{a}_{i3} = \tilde{a}_{23} \cosh \tilde{\theta} - \tilde{a}_{33} \sinh \tilde{\theta}.
\] (3.11)

Similarly, if we choose the timelike ruled surface \( (\tilde{c}_0) \) instead of the spacelike ruled surface \( (\tilde{c}_1) \) then the correspondences of formulae (3.7) and (3.11) are given by
\[
\begin{align*}
\tilde{a}_{02} &= \tilde{a}_{22} \sinh \tilde{\theta} + \tilde{a}_{32} \cosh \tilde{\theta} \\
\tilde{a}_{03} &= -\tilde{a}_{23} \sinh \tilde{\theta} + \tilde{a}_{33} \cosh \tilde{\theta}
\end{align*}
\] (3.12)

respectively, where \( \tilde{\theta} \) is the dual central angle between dual spacelike vectors \( \tilde{a}_{02} \) and \( \tilde{a}_{32} \).

The dual parts \( \theta' \) of dual hyperbolic and central angles are equal to zero because of the normals \( \tilde{a}_{i2} (i=0,1,2,3) \) of ruled surfaces intersection at striction point C. Therefore we take the hyperbolic angle \( \theta \) instead of \( \tilde{\theta} \).

**Proposition 3.1.** The Blaschke trihedrons \([\tilde{a}_{21}, \tilde{a}_{22}, \tilde{a}_{23}]\) and \([\tilde{a}_{31}, \tilde{a}_{32}, \tilde{a}_{33}]\) of the parameter ruled surfaces of congruence \( \tilde{a}(u,v) \) are always coincide such that the order and direction of vectors \( \tilde{a}_{ij} (i=2,3; j=1,2,3) \) is not same. Furthermore the Blaschke derivative formulae are given by
\[
\begin{align*}
\frac{\partial \tilde{a}_{i1}}{\partial \tilde{s}_j} &= \tilde{b}_j \wedge \tilde{a}_{i1} \\
\frac{\partial \tilde{a}_{i2}}{\partial \tilde{s}_j} &= \tilde{b}_j \wedge \tilde{a}_{i2} ; \quad (i, j = 1,2) \\
\frac{\partial \tilde{a}_{i3}}{\partial \tilde{s}_j} &= \tilde{b}_j \wedge \tilde{a}_{i3}.
\end{align*}
\] (3.13)

**Proof.** Writing the vector \( \tilde{a}_0 = \tilde{a}_{22} \wedge \tilde{a}_{32} \) in second equation of (3.1) it is seen that
\[
\begin{align*}
\tilde{a}_{23} &= -(\tilde{a}_{22} \wedge \tilde{a}_{32}) \wedge \tilde{a}_{22} \\
\tilde{a}_{33} &= (\tilde{a}_{22} \wedge \tilde{a}_{32}) \wedge \tilde{a}_{32}
\end{align*}
\]
or
\[
\begin{align*}
\tilde{a}_{23} = - \tilde{a}_{32} \\
\tilde{a}_{33} = \tilde{a}_{22}.
\end{align*}
\]  
(3.14)

By the second equality of (3.12) we have
\[
\frac{\partial \tilde{a}_{33}}{\partial \tilde{s}_3} = \tilde{b}_3 \wedge \tilde{a}_{33}
\]
or
\[
\frac{\partial \tilde{a}_{22}}{\partial \tilde{s}_3} = \tilde{b}_3 \wedge \tilde{a}_{22}.
\]

Similarly it can be shown that other equalities are valid.

Corollary 3.2. The Blaschke vectors \( \tilde{b}_2 \) and \( \tilde{b}_3 \) can be written in the from of
\[
\begin{align*}
\tilde{b}_2 = - \tilde{q}_2 \tilde{a}_0 - \tilde{p}_2 \tilde{a}_{32} \\
\tilde{b}_3 = \tilde{q}_3 \tilde{a}_0 - \tilde{p}_3 \tilde{a}_{22},
\end{align*}
\]  
(3.15)

with \( \tilde{a}_0 \) and the normals \( \tilde{a}_{22} \) and \( \tilde{a}_{32} \) of parameter ruled surfaces.

Proof. By (3.2) we know that
\[
\begin{align*}
\tilde{b}_2 &= - \tilde{q}_2 \tilde{a}_{21} + \tilde{p}_2 \tilde{a}_{23} \\
\tilde{b}_3 &= \tilde{q}_3 \tilde{a}_{13} - \tilde{p}_3 \tilde{a}_{33}.
\end{align*}
\]
If we consider the equalities (3.14) the proof is completed.

Proposition 3.3. Let \( \tilde{s}_i \) (\( i = 1,2,3 \)) be dual arc lengths of ruled surface \( (\tilde{c}_i) \) passing through a straight line of spacelike congruence \( \tilde{a}(u,v) \). If the hyperbolic angle between the normals \( \tilde{a}_{i2} \) and \( \tilde{a}_{22} \) is \( \theta \) then we have
\[
\begin{align*}
\frac{d\tilde{a}_{22}}{d\tilde{s}_1} &= \tilde{b} \wedge \tilde{a}_{22} \\
\frac{d\tilde{a}_{32}}{d\tilde{s}_1} &= \tilde{b} \wedge \tilde{a}_{32} \\
\frac{d\tilde{a}_0}{d\tilde{s}_1} &= \tilde{b} \wedge \tilde{a}_0
\end{align*}
\]  
(3.16)

with
\[
\tilde{b} = \tilde{b}_2 \cosh \theta + \tilde{b}_3 \sinh \theta.
\]  
(3.17)

Proof. We see that the vectors \( \tilde{a}_{i2} (i = 1,2,3) \) are the functions of \( \tilde{s}_2 \) and \( \tilde{s}_3 \). Thus we can write
\[
\frac{d\vec{a}_{22}}{d\vec{s}_1} = \frac{d\vec{a}_{22}}{d\vec{s}_2} \frac{d\vec{s}_2}{d\vec{s}_1} + \frac{d\vec{a}_{22}}{d\vec{s}_3} \frac{d\vec{s}_3}{d\vec{s}_1}.
\]

From the equalities (3.10) and (3.14) we have
\[
\frac{d\vec{a}_{22}}{d\vec{s}_1} = (\vec{b}_2 \wedge \vec{a}_{22}) \cosh \theta + (\vec{b}_3 \wedge \vec{a}_{22}) \sinh \theta
\]
\[
= \vec{b} \wedge \vec{a}_{22}.
\]

Similarly the others are shown.

\[\square\]

**Corollary 3.4.** In proposition 3.3. if we take timelike ruled surface \((\vec{c}_0)\) instead of \((\vec{c}_1)\) then the vector (3.17) is written in the form of
\[
\vec{b}_0 = b_2 \sinh \theta + b_3 \cosh \theta.
\]

**Proof.** It is similar to Prop. 3.3.

**Theorem 3.5.** Let \(\vec{b}_i\) \((i = 1, 2, 3)\) be Blaschke vectors of ruled surfaces \((\vec{c}_i)\) passing through the common spacelike line \(\vec{a}_0\) of spacelike congruence \(\vec{a}(u, v)\). If \(\theta\) is hyperbolic angle between the normals \(\vec{a}_{22}\) and \(\vec{a}_{32}\) of the parameter ruled surfaces then the Blaschke vector of spacelike ruled surface \((\vec{c}_1)\) is given by
\[
\vec{b}_1 = \vec{b}_2 \cosh \theta + \vec{b}_3 \sinh \theta + \vec{a}_0 \frac{d\theta}{d\vec{s}_1},
\]
where \(d\vec{s}_1\) is the dual arc element of \((\vec{c}_1)\).

**Proof.** Differentiating the equation (3.7) with respect to \(\vec{s}_1\) we obtain
\[
\frac{d\vec{a}_{12}}{d\vec{s}_1} = \frac{d\vec{a}_{22}}{d\vec{s}_1} \cosh \theta + \frac{d\vec{a}_{32}}{d\vec{s}_1} \sinh \theta + \vec{a}_{22} \sinh \theta \frac{d\theta}{d\vec{s}_1} + \vec{a}_{32} \cosh \theta \frac{d\theta}{d\vec{s}_1}.
\]

By (3.10) and (3.14) we have
\[
\frac{d\tilde{a}_{12}}{d\tilde{s}_1} = (\tilde{b} \wedge \tilde{a}_{22}) \cosh \theta + (\tilde{b} \wedge \tilde{a}_{32}) \sinh \theta - (\tilde{a}_{32} \wedge \tilde{a}_0) \sinh \theta \frac{d\theta}{d\tilde{s}_1} \\
- (\tilde{a}_{22} \wedge \tilde{a}_0) \sinh \theta \frac{d\theta}{d\tilde{s}_1} \\
= \tilde{b} \wedge (\tilde{a}_{22} \cosh \theta + \tilde{a}_{32} \sinh \theta) - (\tilde{a}_{22} \cosh \theta + \tilde{a}_{32} \sinh \theta) \wedge \tilde{a}_0 \frac{d\theta}{d\tilde{s}_1} \\
= \left(\tilde{b} + \tilde{a}_0 \frac{d\theta}{d\tilde{s}_1}\right) \wedge (\tilde{a}_{22} \cosh \theta + \tilde{a}_{32} \sinh \theta) \\
= \left(\tilde{b} + \tilde{a}_0 \frac{d\theta}{d\tilde{s}_1}\right) \wedge \tilde{a}_{12}.
\]

Denoting the expression \(\tilde{b} + \tilde{a}_0 \frac{d\theta}{d\tilde{s}_1}\) by \(\tilde{b}^*\), we write

\[
\frac{d\tilde{a}_{12}}{d\tilde{s}_1} = \tilde{b}^* \wedge \tilde{a}_{12}. \tag{3.20}
\]

By the last equation of (3.16) we have

\[
\frac{d\tilde{a}_0}{d\tilde{s}_1} = \tilde{b} \wedge \tilde{a}_0 \\
= \left(\tilde{b}^* - \tilde{a}_0 \frac{d\theta}{d\tilde{s}_1}\right) \wedge \tilde{a}_0 \\
= \tilde{b}^* \wedge \tilde{a}_0 \\
\frac{d\tilde{a}_0}{d\tilde{s}_1} = \tilde{b}^* \wedge \tilde{a}_0. \tag{3.21}
\]

By (3.13) and (3.20) we obtain

\[
\tilde{b}^* \wedge \tilde{a}_{12} - \tilde{b}_1 \wedge \tilde{a}_{12} = 0 \\
(\tilde{b}^* - \tilde{b}_1) \wedge \tilde{a}_{12} = 0 \\
\tilde{b}^* - \tilde{b}_1 = \tilde{\lambda} \tilde{a}_{12}; \quad \tilde{\lambda} \in D \tag{3.22}
\]

and

\[
\tilde{b}^* \wedge \tilde{a}_0 - \tilde{b}_1 \wedge \tilde{a}_0 = 0 \\
(\tilde{b}^* - \tilde{b}_1) \wedge \tilde{a}_0 = 0 \\
\tilde{b}^* - \tilde{b}_1 = \tilde{\mu} \tilde{a}_0; \quad \tilde{\mu} \in D \tag{3.23}
\]

By (3.22) and (3.23) we have

\[
\tilde{\lambda} \tilde{a}_{12} = \tilde{\mu} \tilde{a}_0.
\]
and
\[ \tilde{\lambda} = \tilde{\mu} = 0. \]
Consequently it is seen that
\[ \tilde{b}^* - \tilde{b}_1 = 0 \]
or
\[ \tilde{b}^* = \tilde{b}_1. \]
That is, we have
\[ \tilde{b}_1 = \mathbf{b} + \tilde{a}_0 \frac{d\theta}{d\tilde{s}_1}. \]
which completes the proof. \(\square\)

The Balschke vector \(\tilde{b}_1\) has two components. First component coincides with the instantaneous rotation vector \(\tilde{\mathbf{b}}\) of the trihedron \([\tilde{\mathbf{a}}_0, \tilde{\mathbf{a}}_{22}, \tilde{\mathbf{a}}_{32}]\). The other is equal to \(\frac{d\theta}{d\tilde{s}_1}\) which is the component in the direction of normal of the spacelike congruence \(\tilde{a}(u,v)\). Therefore, if a spacelike line on the spacelike congruence move through a spacelike (or timelike) ruled surface then the Blaschke trihedron \([\tilde{a}_0 = \tilde{a}_{11}, \tilde{a}_{22}, \tilde{a}_{32}]\) rotates with respect to trihedron \([\tilde{a}^*, \tilde{a}_{22}, \tilde{a}_{32}]\) with the hyperbolic angular velocity \(\frac{d\theta}{d\tilde{s}_1}\), around the generator, in all instant.

If the hyperbolic angle between \(\tilde{a}_{12}\) and \(\tilde{a}_{22}\) is always constant then we have
\[ \tilde{b}_1 = \tilde{\mathbf{b}}. \]

The real and dual parts of the formula (3.19) are given by
\[ b_1 = b_2 \cosh \theta + b_3 \sinh \theta + a_0 \frac{d\theta}{ds_1} \]
\[ (3.24) \]
and
\[ b_1^* = b_2^* \cosh \theta + b_3^* \sinh \theta + a_0^* \frac{d\theta}{ds_1} - a_0 \frac{d\theta}{ds_1} \frac{ds_1^*}{ds_1}, \]
\[ (3.25) \]
respectively.

**Theorem 3.6.** In Theorem 3.5, if we choose the timelike ruled surface \((\tilde{c}_0)\) instead of \((\tilde{c}_1)\) then the Blaschke vector of \((\tilde{c}_0)\) is given by
\[ \tilde{b}_0 = \tilde{b}_2 \sinh \theta + \tilde{b}_3 \cosh \theta + \tilde{a}_0 \frac{d\theta}{ds_0}, \quad (3.26) \]

where \(\theta\) is the central angle between the normals of surface \((\tilde{c}_0)\) and \((\tilde{c}_1)\), and \(ds_0\) is the arc distance of \((\tilde{c}_0)\).

**Proof.** It is similar to proof of theorem 3.5.

The Blaschke vector \(\tilde{b}_0\) has also two components. First component coincides with the instantaneous rotation vector \(\tilde{b}_0\) of the trihedron \([\tilde{a}_0, \tilde{a}_{22}, \tilde{a}_{32}]\).

The other is equal to \(\frac{d\theta}{ds_0}\), which is the component in the direction of normal of the spacelike congruence \(\tilde{a}(u, v)\). Therefore, if a spacelike line on the spacelike congruence moves through a spacelike (or timelike) ruled surface then the Blaschke trihedron \([\tilde{a}_0, \tilde{a}_{01}, \tilde{a}_{02}, \tilde{a}_{03}]\) rotates with respect to trihedron \([\tilde{a}_0, \tilde{a}_{22}, \tilde{a}_{32}]\) with the central angular velocity \(\frac{d\theta}{ds_0}\), around the generator, in all instant.

If the central angle between \(\tilde{a}_{02}\) and \(\tilde{a}_{22}\) is always constant then we have \(\tilde{b}_0 = \tilde{b}_0\).

The real and dual parts of the formula (3.26) are given by

\[ b_0 = b_2 \sinh \theta + b_3 \cosh \theta + a_0 \frac{d\theta}{ds_0}, \quad (3.27) \]

and

\[ b_0^* = b_2^* \sinh \theta + b_3^* \cosh \theta + a_0^* \frac{d\theta}{ds_0} - a_0 \frac{d\theta}{ds_0} \frac{ds_0^*}{ds_0}, \quad (3.28) \]

respectively.

Now we can give some corollaries for the formulae (3.19) and (3.26):

**Corollary 3.7. (Manheim's Formula)**

There exists the relation

\[ \tilde{p}_1 = \tilde{p}_2 \cosh^2 \theta - \tilde{p}_3 \sinh^2 \theta \quad (3.29) \]

between the dual curvatures of ruled surfaces \((\tilde{c}_1), (\tilde{c}_2)\) and \((\tilde{c}_3)\).

**Proof.** Writing the Blaschke vectors \(\tilde{b}_i (i = 1, 2, 3)\) in the equation (3.19) and multiplying both side of the equation by \(\tilde{a}_{13}\), desired equation is obtained. \(\square\)
The formula (3.29) is valid for all spacelike ruled surfaces passing through a line \( \tilde{a}_0 \) of the spacelike congruence \( \tilde{a}(u,v) \).

The real and dual parts of the formula are as follows:

\[
    p_i = p_i^2 \cosh^2 \theta - p_i^3 \sinh^2 \theta, \quad p_i^* = p_i^2 \cosh^2 \theta - p_i^3 \sinh^2 \theta.
\]  

(3.30)  

(3.31)

**Corollary 3.8. (Liouville’s Formula)**

There exists the relation

\[
    \tilde{q}_1 = \tilde{q}_2 \cosh \theta - \tilde{q}_3 \sinh \theta - \frac{d\theta}{ds_1},
\]  

(3.32)

between the dual torsions of ruled surfaces \((\tilde{c}_1)\), \((\tilde{c}_2)\) and \((\tilde{c}_3)\), where \(ds_1\) is the dual arc element of \((\tilde{c}_1)\) and \(\theta\) is hyperbolic angle between the vectors \((\tilde{a}_{12})\) and \((\tilde{a}_{22})\).

**Proof.** Writing the Blaschke vectors \(\tilde{b}_i\) \((i = 1,2,3)\) in equation (3.18) and multiplying two side of equation by \(\tilde{a}_{11}\), desired equation is obtained.

The real and dual parts of the formula are as follows:

\[
    q_1 = q_2 \cosh \theta - q_3 \sinh \theta - \frac{d\theta}{ds_1},
\]  

(3.33)

\[
    q_1^* = q_2^* \cosh \theta - q_3^* \sinh \theta + \frac{d\theta}{ds_1} \frac{ds_1^*}{ds_1}.
\]  

(3.34)

Let us consider the time like ruled surface \((\tilde{c}_0)\) instead of \((\tilde{c}_1)\). Then we can give:

**Corollary 3.9. (Manheim’s and Liouville’s Formulae)**

The relations between dual curvatures, and dual torsions are given by

\[
    \tilde{p}_0 = - \tilde{p}_2 \sinh^2 \theta + \tilde{p}_3 \cosh^2 \theta
\]  

(3.35)

and

\[
    \tilde{q}_0 = - \tilde{q}_2 \sinh \theta + \tilde{q}_3 \cosh \theta + \frac{d\theta}{ds_0},
\]  

(3.36)

respectively, where \(\theta\) is central angle between \(\tilde{a}_{02}\) and \(\tilde{a}_{32}\).

**Proof.** It is similar to proof of corollaries (3.8) and (3.9).

The formula (3.35) is valid for all timelike ruled surfaces passing through a line \(\tilde{a}_0\) of the spacelike congruence \(\tilde{a}(u,v)\).
The real and dual parts of the formulae (3.35) and (3.36) are given by

\[ p_0 = -p_2 \sinh^2 \theta + p_3 \cosh^2 \theta, \tag{3.37} \]
\[ p_0^* = -p_2^* \sinh^2 \theta + p_3^* \cosh^2 \theta. \tag{3.38} \]

and

\[ q_0 = -q_2 \sinh \theta + q_3 \cosh \theta + \frac{d\theta}{ds_0}, \tag{3.39} \]
\[ q_0^* = -q_2^* \sinh \theta + q_3^* \cosh \theta - \frac{d\theta}{ds_0} \frac{ds_0}{ds_0}, \tag{3.40} \]

respectively.

**Corollary 3.10.** There exists the relation

\[ \tilde{p}_1 + \tilde{p}_0 = \tilde{p}_2 + \tilde{p}_3 \]

between the dual curvatures of arbitrary ruled surfaces \( \tilde{c}_i \) \((i = 0,1)\) and parameter ruled surfaces \( \tilde{c}_j \) \( (j = 2,3) \) of the spacelike congruence \( \tilde{a}(u,v) \).

**Proof.** It is clear. \qed

**Corollary 3.11.** The dual torsions of the parameter ruled surfaces \( \tilde{c}_2 \) and \( \tilde{c}_3 \) are given by

\[ \tilde{q}_2 = \left( \tilde{q}_1 + \frac{d\theta}{ds_1} \right) \cosh \theta + \left( \tilde{q}_0 - \frac{d\theta}{ds_0} \right) \sinh \theta, \tag{3.41} \]

and

\[ \tilde{q}_3 = \left( \tilde{q}_1 + \frac{d\theta}{ds_1} \right) \sinh \theta + \left( \tilde{q}_0 - \frac{d\theta}{ds_0} \right) \cosh \theta, \tag{3.42} \]

respectively.

**Proof.** Multiplying the Liouville's formulae by \( \cosh \theta \) and \( \sinh \theta \), desired equations is obtained. \qed

**Corollary 3.12.** (Hamilton's Formula)

There exits the relation

\[ \tilde{p}_1 = \frac{\tilde{p}_2 - \tilde{p}_3}{2} \sinh 2\theta \tag{3.43} \]
between dual curvatures of ruled surface \( \overline{c}_i \) \( (i=1,2,3) \) of the spacelike congruence \( \overline{a}(\mu,\nu) \).

**Proof.** In equation (3.26) if we take the Blaschke vector \( \overline{b}_i \) instead of \( \overline{b}_0 \) and multiply the two side of equation by \( \overline{a}_{i3} \), the proof is completed.

In the equation (3.43) if \( \overline{p}_2 = \overline{p}_3 \) then we have \( \overline{p}_1 = 0 \). This means that \( \overline{a}(\mu,\nu) \) is a lightlike (null) congruence.

**REFERENCES**


DEGREE OF APPROXIMATION BY NEWLY DEFINED GENERALIZED NÖRLUND MEANS

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ABSTRACT

In the present paper, we obtain the degree of approximation of functions belonging to Lip (ψ(t),u), u > 1 class using the newly defined generalized Nörlund means of the Fourier series f(x). It can be remarked that the order of approximation arrived at is free from the mean generating sequence.

KEYWORDS

Lip (ψ(t),u) class, Nörlund means, Euler-Knopp method, Lip(β,u) class, degree of approximation, Bounded Variation, Hölder's inequality.

AMS SUBJECT CLASSIFICATION: 41A25

1. INTRODUCTION AND RESULTS

1.1. Let f be periodic with period 2π, and integrable in the sense of Lebesgue. The Fourier series associated with f at the point x is given by

\[ f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1.1) \]

Let

\[ S_n(x) = \frac{1}{2} a_0 + \sum_{v=1}^{n} (a_v \cos vx + b_v \sin vx) \quad (1.2) \]

denote the n-th partial sum of the Fourier series (1.1) and

\[ \Phi(t) = \frac{1}{2} \{ f(x + t) + f(x - t) - 2f(x) \}. \]
We say that \( f(x) \in \text{Lip}(\beta, u), u > 1 \) if
\[
\left\{ \int_{-\pi}^{\pi} |f(x + t) - f(x)| \, dx \right\}^{\frac{1}{u}} = O(t^\beta), 0 < \beta \leq 1.
\]
A \( 2\pi \)-periodic function \( f(x) \) is said to belong to the class \( \text{Lip}(\psi(t), u), u > 1 \), if
\[
|f(x + t) - f(x)| \leq M(\psi(t)t^{-\alpha}), 0 < t < \pi,
\]
uniformly in \( x \), where \( \psi(t) \) is a positive increasing function and \( M \) is a positive number independent of \( x \) and \( t \).

If \( f \in C_{2\pi} \), and \( \lim_{t \to 0^+} \psi(t)t^{-\alpha} = 0 \) for given \( \psi \) and \( u > 1 \), then \( \text{Lip}(\psi(t), u) \subset C_{2\pi} \).

1.2. Let \( \{p_n\}, \{q_n\} \) be non-negative, non-increasing generating sequences for the \( (N, p_n, q_n) \) method such that
\[
P_n = p_0 + p_1 + p_2 + \ldots + p_n \to \infty, \text{ as } n \to \infty \tag{1.3}
\]
\[
Q_n = q_0 + q_1 + q_2 + \ldots + q_n, \tag{1.4}
\]
\[
R_n = p_0 q_n + p_1 q_{n-1} + p_2 q_{n-2} + \ldots + p_n q_0 \to \infty, \text{ as } n \to \infty \tag{1.5}
\]
and
\[
t^{p,q}_n = \frac{1}{R_n} \sum_{k=0}^{n} r_k S_{n-k} \tag{1.6}
\]
where \( r_k = p_k q_{n-k}, \quad p_{-1} = q_{-1} = r_{-1} = 0. \)

The method \( (N, p_n, q_n) \) reduces to the following methods:

(i) if \( q_n = 1 \), then \( (N, p_n, q_n) \) method reduces to \( (N, p_n) \) method.

(ii) if \( p_n = 1 \), then \( (N, p_n, q_n) \) method reduces to \( (N, q_n) \) method.

(iii) if \( p_n = \frac{\alpha^n}{n!} \) and \( q_n = \frac{\alpha^n}{n!} \), (where \( \alpha > 0, \delta > 0 \)), then \( (N, p_n, q_n) \) method reduces to Euler-Knopp\( (E, \delta) \) method.

(iv) if \( p_n = \frac{n + \alpha - 1}{\alpha} \) and \( q_n = \frac{n + \beta}{\beta} \), then \( (N, p_n, q_n) \) method reduces to \( (C, \sigma, \beta) \) method.

We write
\[
\Delta p_n = p_n - p_{n-1}
\]
then
\[ R_n = \sum_{k=0}^{n} p_{n-k} q_k = \sum_{k=0}^{n} \Delta p_{n-k} Q_k \]
\[ = \sum_{k=0}^{n} (p_{n-k} - p_{n-k-1})(\sum_{v=0}^{k} (Q_v - Q_{v-1})S_v) \]
\[ = \sum_{k=0}^{n} \Delta p_{n-k} U_k Q_k \]

where
\[ U_k = \frac{1}{Q_k} \sum_{v=0}^{k} (Q_v - Q_{v-1})S_v. \]

Rewriting (1.6) in terms of the simplification given above, we have
\[ t_n^{p,q} = \frac{1}{R_n} \sum_{k=0}^{n} r_k S_{n-k} = \sum_{k=0}^{n} \frac{(p_{n-k} - p_{n-k-1})U_k Q_k}{(p_{n-k} - p_{n-k-1})Q_k} \]
\[ = \sum_{k=0}^{n} \frac{\Delta p_{n-k} U_k Q_k}{\Delta p_{n-k} Q_k}. \]

(1.7)

For any value of the sequence \( \{\Delta p_k\} \), we write
\[ t_n^{(\alpha)p,q} = \frac{\sum_{k=0}^{n} \Delta p_{n-k} U_k^\alpha \lambda_k}{\sum_{k=0}^{n} \Delta p_{n-k} \lambda_k} \]
\[ (1.8) \]

where
\[ U_n^\alpha = \frac{1}{Q_n^\alpha} \sum_{k=0}^{n} (Q_k - Q_{k-1})^\alpha S_k \]
\[ (1.9) \]

\[ Q_n^\alpha = \sum_{k=0}^{n} (Q_k - Q_{k-1})^\alpha, \quad \lambda_k = Q_k^\alpha, \text{ where } \alpha \text{ is a constant.} \]

Now if \( t_n^{(\alpha)p,q} \to s \), as \( n \to \infty \), we say that \( \sum_{n=0}^{\infty} a_n \) is summable \( G(N, p, q) \) method. If \( t_n^{(\alpha)p,q} \in BV \), then \( \sum_{n=0}^{\infty} a_n \) is summable \( \|G(N, p, q)\| \).

Sahney, B.N. and Rao, G. [2] proved a theorem concerning the degree of approximation of functions belonging to \( Lip(\beta, u) \) class for the Nörlund operator \( (N, P_n) \). Their theorem states as follows:
Theorem S [2]: Let \( f(x) \) be periodic and belongs to the class \( \text{Lip}(\beta,u), 0 < \beta \leq 1 \), and let \( \{p_n\} \) be defined as (1.3) and

\[
\left[ \int_a^b \left[ \frac{P(y)}{y^{\beta + 1 - w}} \right]^w dy \right]^{\frac{1}{w}} = O\left( P(n) n^{\beta - \frac{1}{w} - 1} \right)
\]

then

\[
E_n^*(f) = \min_{t_n} \|f - t_n\| = O\left( \frac{1}{n^{\beta - \frac{1}{u}}} \right)
\]

where

\[
t_n = \frac{1}{P_n} \sum_{k=0}^n p_{n-k} S_k.
\]

Khan, Huzoor H. [1] generalized the theorem of Sahney and Rao for the \((N, p_n, q_n)\) means of the Fourier series for the functions belonging to the class \( \text{Lip}(\beta,u) \). The proof given by Sahney and Rao was wrong. Khan Huzoor H. gave the correct proof of the theorem. His theorem states as follows:

Theorem H [1]: If \( f(x) \) is periodic function and belongs to class \( \text{Lip}(\beta,u) \) for \( 0 < \beta \leq 1 \) and if the sequences \( \{p_n\} \) and \( \{q_n\} \) are defined by (1.3) and (1.4) respectively and \( \frac{R(y)}{y^\beta} \) is non-decreasing, then

\[
E_n^*(f) = \min_{t_n^{pq}} \|f - t_n^{pq}\| = O\left( \frac{1}{n^{\beta - \frac{1}{u}}} \right)
\]

where

\[
t_n^{pq} = \frac{1}{R_n} \sum_{k=0}^n p_k q_{n-k} S_k.
\]

The main aim of this note is to generalize the result of Khan, Huzoor H., Sahney and Rao for the newly defined generalized Nörlund means \( G(N, p, q) \) for the functions belonging to the class \( \text{Lip}(\psi(t),u), u > 1 \). It may be remarked that the order of approximation arrived at is free from the mean generating sequences. Our theorem states as follows:

Theorem: The degree of approximation of a periodic function \( f \) with period \( 2\pi \) and belonging to the class \( \text{Lip}(\psi(t),u), u > 1 \) is given by
\[
\max_{0 \leq x \leq 2\pi} \left| f(x) - i_n^{(a)p,q}(x) \right| = O\left[ \sum_{k=1}^{n} k^{-1/\nu} \psi\left(\frac{1}{k}\right) \right]
\]

where \( i_n^{(a)p,q} \) are the generalized \( G(N, p, q) \) means of its Fourier series and the sequences \( \{p_n\} \) and \( \{q_n\} \) are non-negative and non-decreasing, provided \( \psi(t) \) is a positive increasing function such that

\[
\left\{ \int_{0}^{\frac{\pi}{n}} \left( \frac{\psi(t)}{t} \right)^{\frac{1}{\nu}} e^{it} dt \right\}^{\frac{1}{\nu}} = O(\psi\left(\frac{1}{n}\right)n)
\]

and \( \{\psi\left(\frac{1}{n}\right)n^{\frac{1}{\nu}}\}^{\infty}_{n=1} \) is a decreasing sequence.

In order to prove the theorem, we will use the following Lemma:

**Lemma [1]**: If \( \{p_n\} \) and \( \{q_n\} \) are non-negative and non-decreasing then for \( 0 \leq a \leq b \leq \infty, 0 \leq t \leq \pi \) and any \( n \), we have

\[
\left| \sum_{k=a}^{b} p_k q_{n-k} e^{i(n-k)t} \right| \leq R, \quad \text{for any} \; a
\]

where \( R = [\frac{1}{\nu}] \).

**2. PROOF OF THE THEOREM**

We know that

\[
S_n(x) - f(x) = \frac{1}{2\pi} \int_{0}^{\pi} \Phi(t) \frac{\sin(n + \frac{1}{2})t}{\sin \frac{t}{\nu}} dt
\]

and

\[
i_n^{(a)p,q}(x) - f(x) = \frac{1}{2\pi \sum_{n=0}^{\infty} \Delta p_{n-1} \lambda_n} \int_{0}^{\pi} \Phi(t) \left[ \sum_{k=0}^{n} \left( \sum_{v=0}^{k} \Delta p_{n-k} \lambda_v \frac{\sin(k + \frac{1}{2})t}{\sin \frac{t}{\nu}} \right) \right] dt
\]

\[= \frac{1}{2\pi \sum_{n=0}^{\infty} \Delta p_{n-1} \lambda_n} \left[ \int_{0}^{\pi} \Phi(t) \sum_{k=0}^{n} \left( \sum_{v=0}^{k} \Delta p_{n-k} \lambda_v \frac{\sin(k + \frac{1}{2})t}{\sin \frac{t}{\nu}} \right) \right] dt
\]

\[= I_1(x) + I_2(x), \quad \text{(say)}.
\]

For evaluating \( I_1(x) \), we notice that
\[ |I_1(x)| \leq \frac{1}{2\pi \sum_{v=0}^{n} \Delta P_{n-v}} \int_{0}^{\pi/n} \left| \frac{\Phi(t)}{\sin \frac{t}{2}} \right| \left| \sum_{k=0}^{n} \sum_{v=0}^{k} \Delta P_{k-v} \sin(k + \frac{1}{2}t) \right| dt \]

Applying Hölder’s inequality (in the above expression) and using the fact that \( \Phi(t) \in Lip(\psi(t), u) \), \( u > 1 \), we get

\[
|I_1(x)| \leq \frac{1}{\pi \sum_{v=0}^{n} \Delta P_{n-v}} \left[ \left\{ \int_{0}^{\pi/n} \left| \frac{\Phi(t)}{t} \right| \psi(t)^{v} \right. dt \right]^{\frac{1}{v}} \left[ \left\{ \int_{0}^{\pi/n} \left| \sum_{k=0}^{n} \sum_{v=0}^{k} \Delta P_{k-v} \sin(k + \frac{1}{2}t) \right| dt \right]^{\frac{1}{t}}
\]

\[
= O\left[ \left( \int_{0}^{\pi/n} \left| \frac{\psi(t)}{t} \right| \right)^{\frac{1}{v}} \right] O\left( \sum_{v=0}^{n} \Delta P_{n-v} \right) \left( \int_{0}^{\pi/n} dt \right)^{\frac{1}{t}}
\]

\[
= O\left( \psi\left( \frac{1}{n} \right) n^{\frac{1}{v}} \right), \quad \text{(using condition (1.10))}
\]

\[
= O\left( \psi\left( \frac{1}{n} \right) \right)
\]

\[
= O\left( \sum_{k=1}^{n} k^{-1/w} \psi\left( \frac{1}{k} \right) \right), \quad \text{where} \quad w > 1 \quad \text{and} \quad \frac{1}{u} + \frac{1}{w} = 1.
\]

Because,

\[
\psi\left( \frac{1}{n} \right) n^{\frac{1}{v}} = n \psi\left( \frac{1}{n} \right) n^{\frac{1}{w}} = \sum_{k=1}^{n} n^{-1/w} \psi\left( \frac{1}{k} \right)
\]

\[
\leq \left( \sum_{k=1}^{n} k^{-1/w} \psi\left( \frac{1}{k} \right) \right)
\]

we have,

\[
|I_1(x)| \leq \frac{1}{\pi \sum_{v=0}^{n} \Delta P_{n-v}} \int_{0}^{\pi/n} \left| \Phi(t) \right| \left[ \left| \sum_{k=0}^{n} \sum_{v=0}^{k} \Delta P_{k-v} \sin kt \right| \right] dt
\]

\[
+ \frac{1}{\pi \sum_{v=0}^{n} \Delta P_{n-v}} \int_{0}^{\pi/n} \left| \Phi(t) \right| \left[ \left| \sum_{k=0}^{n} \sum_{v=0}^{k} \Delta P_{k-v} \cos kt \right| \right] dt
\]

\[
= I_{1,1}(x) + I_{1,2}(x), (\text{say}).
\]

Applying Hölder’s inequality in \( I_{1,1}(x) \) and using the fact that \( \Phi(t) \in Lip(\psi(t), u) \), \( u > 1 \), we have
\[ |I_{2,1}(x)| \leq \frac{1}{\pi} \sum_{n=0}^{\infty} \Delta p_{n} \lambda_{n} \left( \left[ \frac{\Phi(t)}{\tan \frac{t}{2}} \right]^{\frac{1}{w}} \left[ \sum_{k=0}^{n} \left( \sum_{v=0}^{k} \Delta p_{n} \lambda_{v} \sin kt \right) \right]^{\frac{1}{w}} \right) dt \]

\[ = \frac{1}{\pi} \sum_{n=0}^{\infty} \Delta p_{n} \lambda_{n} \left\{ \left[ \frac{\psi(t)}{t} \right]^{\frac{1}{w}} \cdot O\left( \left[ \sum_{k=0}^{n} \left( \sum_{v=0}^{k} \Delta p_{n} \lambda_{v} \right) \right]^{\frac{1}{w}} \right) \right\} \]

\[ = O\left( \frac{1}{n} \sum_{n=0}^{\infty} \Delta p_{n} \lambda_{n} \left( \sum_{k=0}^{n} \Delta p_{n} \lambda_{n} \nu^{n-k} \right) \right) \]

\[ = O\left( \psi\left( \frac{1}{n} \right) \right) \]

\[ = O\left( \sum_{k=1}^{n} k^{-1/w} \psi\left( \frac{1}{k} \right) \right) \]

Similarly, we get

\[ |I_{2,2}(x)| = O\left( \sum_{k=1}^{n} k^{-1/w} \psi\left( \frac{1}{k} \right) \right) \]

Combining all that, we get

\[ \max_{0 \leq x \leq 2\pi} \left| \tilde{f}_{n}^{(a)p,q}(x) - f(x) \right| = O\left( \sum_{k=1}^{n} k^{-1/w} \psi\left( \frac{1}{k} \right) \right) \]

which completes the proof of the theorem.

Using similar techniques, we can prove the following theorem on conjugate Fourier series.

**Theorem:** The degree of approximation of a periodic function \( \tilde{f} \) with period \( 2\pi \) and belonging to the class \( \text{Lip}(\psi(t), u), u > 1 \) is given by

\[ \max_{0 \leq x \leq 2\pi} \left| \tilde{f}(x) - \tilde{f}_{n}^{(a)p,q}(x) \right| = O\left( \sum_{k=1}^{n} k^{-1/w} \psi\left( \frac{1}{k} \right) \right) \]

where \( \tilde{f}_{n}^{(a)p,q}(x) \) is the generalized \( G(N, p, q) \) means of its conjugate Fourier series \( \tilde{f} \).
REFERENCES
