QUARTER – SYMMETRIC METRIC CONNECTION ON A SASAKIAN MANIFOLD

U. C. DE and JOYDEEP SENGUPTA

Department of Mathematics, University of Kalyani.
Kalyani, Nadia, West Bengal, 741235, INDIA

(Received April 27, 1999; Revised Nov. 19, 1999; Accepted Feb. 07, 2000)

ABSTRACT

The object of this paper is to prove the existence of a quarter-symmetric metric connection on a Riemannian manifold and to study some properties of a curvature tensor of a quarter-symmetric metric connection on a Sasakian manifold.

1. INTRODUCTION

Let \((M^n, g)\) be a contact Riemannian manifold with a contact form \(\eta\), the associated vector field \(\xi\), \((1,1)\) tensor field \(\phi\) and the associated Riemannian metric \(g\). If \(\xi\) is a Killing vector field, then \((M^n, g)\) is called a K-contact Riemannian manifold \([2][9]\). A K-contact Riemannian manifold is called Sasakian \([2]\) if

\[
(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X
\]

holds, where \(\nabla\) denotes the operator of covariant differentiation with respect to \(g\).

A linear connection \(\tilde{\nabla}\) on an \(n\)-dimensional Riemannian manifold \((M^n, g)\) is called a quarter-symmetric connection \([4]\) if its torsion tensor \(T\) satisfies

\[
T(X, Y) = \pi(Y)F(X) - \pi(X)F(Y)
\]

where \(\pi\) is a differentiable 1-form and \(F\) is a \((1, 1)\) tensor field. If, moreover, the connection \(\tilde{\nabla}\) satisfies

\[
(\tilde{\nabla}_X g)(Y, Z) = 0
\]

for all vector fields \(X, Y, Z\) on \((M^n, g)\) then it is called a quarter-symmetric metric connection.

Quarter-symmetric metric connection have been studied by K.Yano and T. Imai \([10]\). In this connection we can also mention the works of S.C. Rastogi \([7][8]\).
D. Kamilya and U.C. De \[5\], S.C. Biswas and U.C. De \[6\], R.S. Mishra and S.N. Pandey \[6\], S. Golab \[4\] and others.

If $F(X)=X$, then the connection is called a semi-symmetric metric connection \[11\]. In the present paper we have studied a Sasakian manifold with a quarter-symmetric metric connection $\tilde{\nabla}$ satisfying (1.2) and (1.3) in which the 1-form $\pi$ and the (1, 1) tensor field $F$ are respectively identical with the contact form $\eta$ and the (1, 1) tensor field $\phi$ of the contact structure $(\phi, \xi, \eta, g)$ so that the relation (1.2) takes the form

$$T(X, Y) = \eta(Y)\phi(X) - \eta(X)\phi(Y). \quad (1.4)$$

At first we prove the existence of a quarter-symmetric metric connection in a Riemannian manifold $(M^n, g)$. In section 3 we deduce the expressions for the curvature tensor and the Ricci tensor of $(M^n, g)$ with respect to the quarter-symmetric metric connection. In general, the Ricci tensor of the quarter-symmetric metric connection is not symmetric. Here it is proved that in a Sasakian manifold the Ricci tensor of a quarter-symmetric metric connection is symmetric. Also, in general, the conformal curvature tensors of the quarter-symmetric metric connection and the Riemannian connection are not equal. Here we obtain a necessary and sufficient condition for the conformal curvature tensor to be equal. Finally, we obtain an expression of the projective curvature tensor of the quarter-symmetric metric connection.

2. PRELIMINARIES

Let $R$ and $S$ denote respectively the curvature tensor and Ricci tensor of type (1.2) of $(M^n, g)$. It is known that in a Sasakian manifold $(M^n, g)$ besides the relation (1.1), the following relations hold \[2\]|\[9\]

$$\phi(\xi) = 0 \quad (2.1)$$
$$\eta(\xi) = 1 \quad (2.2)$$
$$\phi^2 X = -X + \eta(X)\xi \quad (2.3)$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (2.4)$$
$$g(\xi, X) = \eta(X) \quad (2.5)$$
$$\nabla_X \xi = -\phi X \quad (2.6)$$
$$S(X, \xi) = (n - 1)\eta(X) \quad (2.7)$$
$$g(R(\xi, X)Y, \xi) = g(X, Y) - \eta(X)\eta(Y) \quad (2.8)$$
$$R(\xi, X)\xi = -X + \eta(X)\xi \quad (2.9)$$
$$\nabla_X \phi(Y) = R(\xi, X)Y \quad (2.10)$$

for any vector fields $X, Y$.

We can define a 2 form $\phi$ in a Sasakian manifold $(M^n, g)$ by

$$\phi(X, Y) = g(X, \phi Y) \quad (2.11)$$
such that
\[ \phi = d\eta. \] (2.12)

From (2.3), (2.4), (2.11) and (2.12) by using \( \eta \cdot \phi = 0 \) we get
\[ g(\phi X, Y) + g(X, \phi Y) = 0 \] (2.13)
\[ d\eta(\phi X, Y) + d\eta(X, \phi Y) = 0 \] (2.14)
\[ d\eta(\phi X, \phi Y) = d\eta(X, Y) \] (2.15)
and
\[ d\eta(\xi, X) = 0 \] (2.16)

3. **EXISTENCE OF A QUARTER-SYMMETRIC METRIC CONNECTION**

Let \( X, Y \) be two vector fields on \((M^n, g)\) We define \( \nabla_X Y \) by the following equation
\[ 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(\{Y, X\} Z) - g(\{X, Z\} Y) + g(\{Z, Y\} X) + g(\pi(X)\phi Z - \pi(Z)\phi X, Y) + g(\pi(Y)\phi Z - \pi(Z)\phi Y, Z) + g(\pi(X)\phi Y - \pi(X)\phi Y, Z) \] (3.1)

which should hold for all vector fields \( Z \) on \((M^n, g)\).

It can be verified that the mapping \((X, Y) \rightarrow \nabla_X Y\) satisfies the following equalities:
(i) \( \nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z \)
(ii) \( \nabla_X,\gamma Z = \nabla_X Z + \nabla_Y Z \)
(iii) \( \nabla_X f(Y) = f\nabla_X Y, \forall f \in F(M^n) \)
(iv) \( \nabla_X f(Y) = f\nabla_X Y + (Xf)Y, \forall f \in F(M^n) \)

where \( F(M^n) \) denotes the set of all differentiable mappings over \( M^n \). Therefore \( \nabla \) determines a linear connection on \((M^n, g)\) Now we have,
\[ 2g(\nabla_X Y, Z) - 2g(\nabla_Y Y, Z) = 2g(\{X, Y\} Z) + 2g(\pi(Y)\phi(X) - \pi(X)\phi(Y), Z). \] (3.2)
Hence,
\[ \nabla_X Y - \nabla_Y X - [X, Y] = \pi(Y)\phi(X) - \pi(X)\phi(Y) \]
or,
\[ T(X, Y) = \pi(Y)\phi(X) - \pi(X)\phi(Y). \] (3.3)

Also we have,
\[ 2g(\nabla_X Y, Z) + 2g(\nabla_Y Y, Z) = 2Xg(Y, Z) \]
or,
\[ (\nabla_X g)(Y, Z) = 0 \]
that is,
\[ \nabla_g = 0. \] (3.4)
From (3.3) and (3.4) it follows that \( \tilde{\nabla} \) determines a quarter-symmetric metric connection on \((M^n,g)\). It can be easily shown that \( \tilde{\nabla} \) determines a unique quarter-symmetric metric connection on \((M^n,g)\) Thus we have the following theorem:

**Theorem 2.1.** Let \((M^n,g)\) be a Riemannian manifold and \( \pi \) be a 1-form on \( M^n \). Then there exists a unique linear connection \( \tilde{\nabla} \) satisfying (3.3) and (3.4).

**Remark.** Theorem (2.1) proves the existence of a quarter-symmetric metric connection on \((M^n,g)\).

4. **CURVATURE TENSOR**

Let us write
\[
\tilde{\nabla}_X Y = \nabla_X Y + H(X,Y)
\]  
(4.1)

For a quarter-symmetric metric connection \( \tilde{\nabla} \) and a Levi-Civita connection \( \nabla \) on \((M^n,g)\).

From (3.4) we get
\[
Xg(Y,Z) - g(\nabla_X Y + H(X,Y),Z) - g(Y,\nabla_X Z + H(X,Z)) = 0.
\]

By virtue of (4.1) we have,
\[
Xg(Y,Z) - g(\tilde{\nabla}_X Y, Z) - g(Y,\tilde{\nabla}_X Z) - g(H(X,Y),Z) - g(H(X,Z),Y) = 0.
\]

From here
\[
Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y,\nabla_X Z) - g(H(X,Y),Z) - g(H(X,Z),Y) = 0
\]

that is,
\[
(\nabla_X g)(Y,Z) - g(H(X,Y),Z) - g(H(X,Z),Y) = 0.
\]

Since \( \nabla \) is the Levi-Civita connection, \( (\nabla_X g)(Y,Z) = 0 \) and hence we have
\[
g(H(X,Y),Z) - g(H(X,Z),Y) = 0.
\]  
(4.2)

Also, from (4.1) it follows that
\[
H(X,Y) - H(Y,X) = \tilde{\nabla}_X Y - \nabla_X Y - \tilde{\nabla}_Y X + \nabla_Y X
\]
\[
= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]
\]
\[
= T(X,Y).
\]

Hence by using (1.4) we obtain,
\[
H(X,Y) - H(Y,X) = \eta(Y)\phi(X) - \eta(Y)\phi(X).
\]  
(4.3)

Again from (4.3) we get
\[
g(H(X,Y),Z) - g(H(Y,X),Z) = \eta(Y)g(\phi(X),Z) - \eta(X)g(\phi(Y),Z)
\]  
(4.4)

\[
g(H(X,Z),Y) - g(H(Z,X),Y) = \eta(Z)g(\phi(X),Y) - \eta(X)g(\phi(Z),Y)
\]  
(4.5)

and
\[
g(H(Y,Z),X) - g(H(Z,Y),X) = \eta(Z)g(\phi(Y),X) - \eta(Y)g(\phi(Z),X)
\]  
(4.6)
Adding (4.4) and (4.5) and then subtracting (4.6) from the result we get by applying (2.13) and (4.2)
\[ H(Z, Y) = -\eta(Z)\phi(Y). \] (4.7)
So that from (4.1) we can write
\[ \widetilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi(Y). \] (4.8)
If
\[ \widetilde{R}(X, Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X, Y]} Z \]
denote the curvature tensor of the connection \( \widetilde{\nabla} \) then from (4.8) by using the relation (1.1) we obtain,
\[ \widetilde{R}(X, Y)Z = R(X, Y)Z - 2d\eta(X, Y)\phi(Z) + \eta(X)g(Y, Z)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - \eta(Y)g(X, Z)\xi \] (4.9)
where \( R(X, Y)Z \) is the Riemannian curvature tensor of the manifold. Therefore from (4.9) we obtain,
\[ \widetilde{S}(Y, Z) = S(Y, Z) - 2d\eta(\phi Z, Y) + g(Y, Z) + (n - 2)\eta(Y)\eta(Z) \] (4.10)
where \( S \) and \( \widetilde{S} \) denote the Ricci tensors of \( \nabla \) and \( \widetilde{\nabla} \) respectively. Further, since \( d\eta \) is skew-symmetric, we get from (4.10)
\[ \widetilde{\tau} = r + 2(n - 1). \] (4.11)
It is seen from (4.10) by using (2.14) that
\[ \widetilde{S}(X, Y) = \widetilde{S}(Y, X) \]
that is, the Ricci tensor of the connection \( \widetilde{\nabla} \) is symmetric.
The conformal curvature tensor \( \widetilde{C}(X, Y)Z \) of the quarter-symmetric metric connection \( \widetilde{\nabla} \) is given by
\[ \widetilde{C}(X, Y)Z = \widetilde{R}(X, Y)Z - \frac{1}{n - 2}[g(Y, Z)\widetilde{L}X - g(X, Z)\widetilde{L}Y + \widetilde{S}(Y, Z)X - \widetilde{S}(X, Z)Y] \]
\[ + \frac{\widetilde{\tau}}{(n - 1)(n - 2)}[g(Y, Z)X - g(X, Z)Y]. \] (4.12)
From which we get
\[ \tilde{C}(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) - \frac{1}{n - 2}[g(Y, Z)\tilde{S}(X, W) - g(X, Z)\tilde{S}(Y, W) \]
\[ + \tilde{S}(Y, Z)g(X, W) - \tilde{S}(X, Z)g(Y, W)] \]
\[ + \frac{\tilde{\tau}}{(n - 1)(n - 2)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \] (4.13)
where
\[ \tilde{C}(X, Y, Z, W) = g(\tilde{C}(X, Y), Z, W) \]
\[ \tilde{R}(X, Y, Z, W) = g(\tilde{R}(X, Y)Z, W) \]
\[ \tilde{S}(X, Y) = g(\tilde{L}X, Y), \]
tilde{L} being the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor $\tilde{S}$.

By using (4.9), (4.10) and (4.11) we get from (4.13)

$$\tilde{C}(X, Y, Z, W) = C(X, Y, Z, W) - 2d\eta(X, Y)g(\phi Z, W)$$
$$+ \frac{2}{n-2}[g(Y, Z)d\eta(\phi W, X) - g(X, Z)d\eta(\phi W, Y)$$
$$+ g(X, W)d\eta(\phi Z, Y) - g(Y, W)d\eta(\phi Z, X)] = 0$$

(4.14)

where $C(X, Y, Z, W) = g(C(X, Y)Z, W)$ and $\tilde{C}(X, Y)Z$ is the conformal curvature tensor of the Levi-Civita connection $\nabla$.

Clearly $d\eta = 0$ is a sufficient condition for


(4.15)

If one considers the relation (4.15) to be true one can get from (4.14)

$$d\eta(X, Y)g(\phi Z, W) - \frac{1}{n-2}[g(Y, Z)d\eta(\phi W, X) - g(X, Z)d\eta(\phi W, Y)$$
$$+ g(X, W)d\eta(\phi Z, Y) - g(Y, W)d\eta(\phi Z, X)] = 0.$$ 

(4.16)

Putting $Z = \xi$ in (4.16) we get

$$\eta(Y)d\eta(\phi W, X) - \eta(X)d\eta(\phi W, Y) = 0.$$ 

(4.17)

Again putting $Y = \xi$ and $W = \phi W$ in (4.17) we obtain

$$d\eta(X, W) = 0.$$ 

The following theorem can now be stated:

**Theorem 4.1.** A necessary and sufficient condition for the conformal curvature tensor of the quarter-symmetric metric connection $\tilde{\nabla}$ given by (4.1) to be equal to the conformal curvature tensor of a Sasakian manifold $M^n$ is that the contact form $\eta$ is closed.

Next we consider the projective curvature tensor of $\tilde{\nabla}$. Let

$$\tilde{P}(X, Y)Z = \tilde{R}(X, Y)Z + \frac{1}{n+1}[\tilde{S}(X, Y)Z - \tilde{S}(Y, X)Z]$$
$$+ \frac{1}{n^2-1} \left[ n\tilde{S}(X, Z) + \tilde{S}(Z, X) \right]$$

$$- \left[ n\tilde{S}(Y, Z) + \tilde{S}(Z, Y) \right]$$

(4.18)

be the generalized projective curvature tensor [3] of the quarter-symmetric metric connection $\tilde{\nabla}$. But here the Ricci tensor of the quarter-symmetric metric connection is symmetric. Therefore the expression of the generalized projective curvature tensor of the connection $\tilde{\nabla}$ reduces to

$$\tilde{P}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{n-1}[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y]$$

(4.19)

From which we get
\[ \tilde{P}(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) - \frac{1}{n-1} \left[ \tilde{S}(Y, Z)g(X, W) - \tilde{S}(X, Z)g(Y, W) \right] \] (4.20)

where

\[ \tilde{P}(X, Y, Z, W) = g(\tilde{P}(X, Y)Z, W) \]
\[ \tilde{R}(X, Y, Z, W) = g(\tilde{R}(X, Y)Z, W). \]

By virtue of (4.9) and (4.10) we get from (4.20)

\[ \tilde{P}(X, Y, Z, W) = P(X, Y, Z, W) - 2\eta(X, Y)g(\phi Z, W) \]
\[ + g(Y, Z)\eta(X)\eta(W) - g(X, Z)\eta(Y)\eta(W) \]
\[ + \frac{2}{n-1} [g(X, W)\eta(\phi Z, Y) - g(Y, W)\eta(\phi Z, X)] \]
\[ - \frac{1}{n-1} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \]
\[ + \frac{1}{n-1} [g(X, Y)\eta(Y)\eta(Z) - g(Y, W)\eta(X)\eta(Z)] \] (4.21)

where \( P(X, Y, Z, W) = g(P(X, Y)Z, W) \) and \( P(X, Y)Z \) is the projective curvature tensor of the Levi-Civita connection. Hence the projective curvature tensor of the quarter-symmetric metric connection given by (4.1) is not, in general, equal to the projective curvature tensor of the Levi-Civita connection.

ACKNOWLEDGEMENT. The authors are thankful to the referee for his valuable suggestions in the improvement of the paper.

REFERENCES