DISCRETE SETS AND IDEALS

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ABSTRACT

In this paper, the discrete sets and corresponding dual ideals and principal maximal ideals in \( B(X) \) are studied, where \( X \) is an \( n \)-dimensional complex manifold and \( B(X) \) is a ring (algebra) of holomorphic functions defined on \( X \).

1. INTRODUCTION

a) Let us denote the open unit disc in \( \mathbb{C} \) by \( U \) and the unit disc bounding \( U \) by \( T \). Similarly, in \( \mathbb{C}^n \), the open unit disc and its boundary are defined by

\[ U^n = \{ z \in \mathbb{C}^n : |z_i| < 1, \ 1 \leq i \leq n \} \]

and

\[ T^n = \{ z \in \mathbb{C}^n : |z_i| = 1, \ 1 \leq i \leq n \} \]

respectively.

\( U^n \) is the cartesian product of \( U \) by itself \( n \) times and \( T^n \) is the cartesian product of \( T \) by itself \( n \) times. For \( n > 1 \), \( T^n \) is a subset of the topological boundary \( \partial U^n \). If \( n=1 \), then \( U'=U \) and \( T'=\partial T \).

b) More generally, an open polydisc in \( \mathbb{C}^n \) is the cartesian product of \( n \) open discs. The polydisc with radius \( r = (r_1, r_2, \ldots, r_n) \) and center \( z^0 = (z_1^0, z_2^0, \ldots, z_n^0) \) is

\[ P_r^n = \{ z \in \mathbb{C}^n : |z_i - z_i^0| < r_i, \ 1 \leq i \leq n \} \]

and the boundary of \( P_r^n \) is defined by
\[ T^n_r = \{ z \in \mathbb{C}^n : |z_i - z_i^0| = r_i, 1 \leq i \leq n \} \]

The closure of \( U^n \) defined by \( \overline{U}^n \). Then \( \overline{U}^n = U^n \cup T^n \). i.e.

\[ \overline{U}^n = \{ z \in \mathbb{C} : |z_i - z_i^0| \leq 1, 1 \leq i \leq n \} \]

The problem of discarding the slower is of great importance in practice, [6].

1.1. Definition. Let \( X \) be a topological space and let \( D \subset X \). If \( D \) has no limit points, then it is called a discrete subset (of \( X \)).

Let \( G \) be a region (open connected set) in \( \mathbb{C} \), and let \( A(G) \) be the ring (or complex algebra) of complex valued analytic functions in \( G \). The set of zeros of \( f \) in \( G \), \( S(f) = \{ z \in G : f(z) = 0 \} \), for \( f \in A(G) \), is a discrete set.

Here \( S(f) \) is thought algebraically. That is, the zeros are counted by multiplicity in \( S(f) \) and also in the union and intersection. If \( K \) is a subset of \( A(G) \), then \( S(K) = \bigcup_{f \in K} S(f) \). The following lemmas are well-known from [3].

1.2. Lemma. Let \( \{ x_k \}_{k=1}^\infty \) be a discrete sequence, \( \{ m_k \} \) be a discrete sequence of positive integers and \( \{ \beta_{k,p} : p = 0,1,\ldots,m_{k-1} : k = 1,2,\ldots \} \) be a sequence of complex numbers. Then there exists an \( f \in A(G) \) so that \( f^{(p)}(\alpha_k) = \beta_{k,p} \) (\( p = 0,1,\ldots,m_{k-1} : k = 1,2,\ldots \)).

1.3. Lemma. Let \( f_1, f_2 \in A(G) \) and let \( S(f_1) \cap S(f_2) = \emptyset \). Then for every \( h \in A(G) \), there exist \( g_1, g_2 \in A(G) \) so that \( h = f_1 g_1 + f_2 g_2 \).

1.4. Lemma. If \( f_1, f_2 \in A(G) \), then there exists \( g_1, g_2 \in A(G) \) so that \( S(f_1 g_1 + f_2 g_2) = S(f_1) \cap S(f_2) \).

2. Dual Ideals

Let \( I \) be an ideal of \( A(G) \). If there exists a point \( z_0 \in G \) so that \( f(z_0) = 0 \) for every \( f \in I \), then \( I \) is called an ideal of type I, and in general it is denoted by \( I_{z_0} \). Then

\[ I_{z_0} = \{ f \in A(G) : f(z_0) = 0 \} \]
Other ideals of \( A(G) \) are called of type II.

2.1. Definition. Let us denote a family of nonempty discrete subsets of \( G \) by \( H \). If the following conditions are satisfied, then \( H \) is called the dual ideal (of \( G \)).

1) If \( D_1, D_2 \in H \) then \( D_1 \cap D_2 \in H \)
2) If \( D_1 \in H \) and \( D_2 \) is a discrete subset of \( G \) such that \( D_1 \subseteq D_2 \), then \( D_2 \in H \).

By Zorn lemma there exists a maximal dual ideal. (Let \( B \) be a dual ideal of \( G \). If there is not a dual ideal \( B' \) of \( B \) so that \( B' \) contains \( B \) as a proper subset then \( B \) is called maximal dual ideal.) If \( B \) is a maximal dual ideal, then there exists a discrete set \( D \in H \) such that \( D \cap D^\prime = \phi \) for every discrete subset \( D' \) not belonging to \( H \).

Let \( B \) be the maximal dual ideal of discrete subsets of \( G \). If there exists a point \( z_0 \in G \) such that \( z_0 \in D \) for every \( D \in H \) then \( B \) is called a maximal dual ideal of type I. All other maximal dual ideals of discrete subsets of \( G \) are called maximal dual ideals of type II.

2.2. Theorem. 1) For every maximal dual ideal \( B \) of discrete subsets of \( G \)
\( I(B) = \{ f, f \in A(G), S(f) \in B \} \) is a maximal dual ideal of \( A(G) \).
2) Conversely, for every maximal ideal \( I \) of \( A(G) \), \( B(I) = \{ S(f) : f \in I \} \) is a maximal dual ideal of discrete subsets of \( G \).
3) Let us denote the set of maximal ideals of \( A(G) \) by \( M \) and the set of maximal dual ideals of discrete subsets of \( G \) by \( N \). Then the maps \( \phi \) and \( \psi \) defined by \( \phi : N \rightarrow M \), \( \phi(B) = I(B) \) and \( \psi : M \rightarrow N \), \( \psi(I(B)) = B \) are one to one and onto. \( B \) is a maximal dual ideal of type I or II according as the corresponding \( I(B) \) is a maximal ideal of type I or II [3].

2.3. Theorem. Let \( R \) be an open Riemann surface, \( A(R) \) be ring of analytic functions defined on \( R \) and \( B \) be a dual ideal of \( R \) then \( I(B) = \{ f \in A(R) : S(f) \in B \} \) is an ideal of \( A(R) \).

Proof. If \( f_1, f_2 \in I(B) \) then \( S(f_1), S(f_2) \in B \). Since \( B \) is a dual ideal \( S(f_1) \cap S(f_2) \subseteq S(f_1 \cdot f_2) \), \( S(f_1 \cdot f_2) \in B \) and therefore \( f_1 \cdot f_2 \in I(B) \).

Let \( f \in I(B) \) and \( g \in A(R) \) be arbitrary. As \( S(f) \in B \) and \( S(f) \subseteq S(fg) \) we have \( S(fg) \in B \).

Then \( fg \in I(B) \) and therefore \( I(B) \) is an ideal of \( A(R) \). Also if \( B_1 \subseteq B_2 \) then \( I(B_1) \subseteq I(B_2) \) is obvious.

2.4. Theorem. \( A_D^1 = \{ f \in A(G) : \text{ for every } z \in D, f'(z) = 0 \} \) is a subring of \( A(G) \) for a discrete subset \( D \) of \( G \). (Here \( f' \) denotes the derivative of \( f \).)
Proof. If $f, g \in A_D^1$ then as $(f' - g')(z) = (f' - g')(z) = 0$ for every $z \in D$, $f - g \in A_D^1$. Similarly as $(f'')(z) = 0$ for every $z \in D$, $A_D^1$ is a subring of $A(G)$.

Corollary. If $A_D^{(n)} = \{ g \in A_D^{(n-1)} : g^{(n)}(z) = 0 \, z \in D, \, n \geq 2 \}$ then $A_D^{(n)}$ is a subring of $A_D^{(n-1)}$. Further $\bigcap_{n=1}^{\infty} A_D^{(n)} = C$.

Proof. If $f \in \bigcap_{n=1}^{\infty} A_D^{(n)}$ then $f^{(n)}(z) = 0$ for $n = 1, 2, \ldots$ ($z \in D$) This implies that $f$ is a constant.

3. COVERING SPACES

3.1. Definition. Let $X$ and $\tilde{X}$ be two topological spaces and let $p: \tilde{X} \to X$ be a continuous map. If the following conditions are satisfied then $\tilde{X}$ is called the covering space of $X$.

1) For every $x \in X$, there exists an open neighbourhood $W$ of $x$ so that $p^{-1}(W)$ is union of some open sets $W_\alpha$ in $\tilde{X}$ ($\alpha \in I$).
2) $p|_{W_\alpha}$ is a local homeomorphism of $W_\alpha$ onto $W$ ($\alpha \in I$).

If $\tilde{X}$ is a covering space of $X$, the map $p$ is called a covering map. If $p(\tilde{X}) = X$ then $X$ is called the projection of $\tilde{X}$.

3.2. Definition. Let $\tilde{X}$ be a covering space of $X$, $p: \tilde{X} \to X$ a covering map and $g: \tilde{X} \to \tilde{X}$ be a homeomorphism. If $p \circ g = p$ i.e. $p(g(\tilde{x})) = p(\tilde{x})$ then $g$ is called a covering map of $\tilde{X}$.

Hence a covering map permutes the points with the same projections. The covering transformations form a group under combination. This group is called the group of covering transformations, [2], [4].

Let $p: \tilde{X} \to X$ be a covering map and $x \in X$ where $X$ is a Hausdorff space. Let $W$ be a neighbourhood of $x$ in the meaning of Definition 3.1. Let us take a neighbourhood $U$ of $x$ so that $\tilde{U} \subset W$. If we form a set $K = \{ k_\alpha \}$ for each $W_\alpha$ where $k_\alpha \in (W_\alpha \cap p^{-1}(U))$ then the following lemma can be given.

3.3. Lemma. $K$ is a discrete set.

Proof. Conversely let us suppose $k$ is a limit point of $K$. Let $V$ be a neighbourhood of $p(k)$. Since $p$ is continuous, there exists a neighbourhood $V_1$ of $k$ so that $p(V_1) \subset V$. Let $k_\alpha \in (V_1 - k) \cap K$ then $p(k_\alpha) \in U$. Hence $V \cap U \neq \emptyset$. That is the
intersection of a neighbourhood of \( p(k) \) with \( U \) is nonempty. Hence \( p(k) \) is a limit point of \( U \). That is \( p(k) \in \overline{U} \). Since \( \overline{U} \subset W \), there exists a \( W_{\alpha} \) so that \( k \in W_{\alpha} \). But there can only be \( k_{\alpha} \) in \( W_{\alpha} \) by hypothesis. Therefore \( k \) can not be a limit point of \( K \).

Notice that if \( \widetilde{X} \) is a covering space of \( X \) and \( p: \widetilde{X} \rightarrow X \) is a covering map then \( p^{-1}(x) \) has a discrete topology for every \( x \in X \). Because the intersection of the open set \( W_\alpha \) with \( p^{-1}(x) \) consist of one point. Therefore this point is open in the subspace topology on \( p^{-1}(x) \). Further for \( x, y \in X \) the cardinalities of \( p^{-1}(x) \) and \( p^{-1}(y) \) are equal.

3.4. Definition. Let \( R \) be a Riemann surface and \( D \) be a discrete subset of \( R \). The ideal \( I_0 = \{ f \in A(R): f(p) = 0, \text{ for } p \in D \} \) is called discrete ideal of \( A(R) \). For \( I_q = \{ f \in A(R): f(q) = 0 \} \) we can give the following theorem.

3.5. Theorem. Let \( R \) and \( \widetilde{R} \) be two Riemann surfaces, \( \widetilde{R} \) be a covering surface of \( R \), \( p: \widetilde{R} \rightarrow R \) be a covering map and \( g: \widetilde{R} \rightarrow \widetilde{R} \) be a covering transformation. Then

a) Let \( A = \{ I_{q_i} : q_i \in p^{-1}(x) \} \) for \( x \in R \). Then the map \( \phi : A \rightarrow A, \phi(q_i) = I_{g(q_i)} \) is one-to-one and onto.

b) Let \( B = \{ I_{p^{-1}(x)} : x \in R \} \). Then \( \psi : R \rightarrow B, \psi(x) = I_{p^{-1}(x)} \) is one-to-one and onto.

Proof. a) First we show that \( \phi \) is a map. If \( I_{q_1} = \{ f \in A(\widetilde{R}): f(q_1) = 0 \} = I_{q_2} = \{ g \in A(\widetilde{R}): g(q_2) = 0 \} \) then there exists \( f \in I_{q_1} \) so that \( S(f) = \{ q_i \} \) by \([1]\) and \( I_{q_1} = \{ f \in I_{q_1} \} \). Since \( f \in I_{q_1} \), \( f(q_2) = 0 \). Then \( q_1 = q_2 \). Therefore since \( g(q_1) = g(q_2) \), \( \phi(I_{q_1}) = \phi(I_{q_2}) \). That is \( \phi \) is a map. If \( \phi(I_{q_1}) = \phi(I_{q_2}) \), then \( I_{g(q_1)} = I_{g(q_2)} \Rightarrow g(q_1) = g(q_2) \Rightarrow q_1 = q_2 \Rightarrow I_{q_1} = I_{q_2} \), i.e. \( \phi \) is one-to-one. Finally let \( I_{q_1} \in A \). Since \( g \) is onto there exists a \( q_i \in p^{-1}(x) \) so that \( g(q_i) = q_i \). Then \( \phi(I_{q_1}) = I_{q_1} \).

b) It is easy to see that \( \psi \) is a map. To show that it is one-to-one let \( \psi(x) = \psi(y) \), i.e. \( I_{p^{-1}(x)} = I_{p^{-1}(y)} \). Then since \( p^{-1}(x) \) is a discrete set, by generalized Weierstrass theorem there exists a \( f \in A(R) \) so that \( S(f) = p^{-1}(x) \) \([5]\). But since \( f \in I_{p^{-1}(y)} \), \( S(f) = p^{-1}(y) \). Let \( x = y \), where \( x \in p^{-1}(x) \) and \( y \in p^{-1}(x) \). Then \( x = p(x_i) = p(y_i) = y \). This shows that \( \psi \) is one-to-one. By the definition \( \psi \) is onto.
4. n-DIMENSIONAL COMPLEX MANIFOLDS

4.1. Definition. Let X be a topological space, U be an open subset of X, and \( \psi \) be a topological map from U to \( \mathbb{C}^n \). The pair \((U, \psi)\) is called coordinate card or card in X. If \( a \in U \) then \((U, \psi)\) is said to contain a.

4.2. Definition. Let X be a connected Hausdorff space and \( \phi = \{(U_i, \psi_i): i \in I\} \) be set of cards in X. If the following conditions are satisfied then \( X=(X, \phi) \) is called an n-Dimensional Complex Manifold.

1) Every \( x \in X \) is in only one card. That is the family \( \{U_i: i \in I\} \) forms an open cover of X.

2) If \((U_1, \psi_1), (U_2, \psi_2) \in \phi \) and \( U_1 \cap U_2 \neq \phi \) then

\[
\psi_{12} = \psi_1 \circ \psi_2^{-1}: \psi_2(U_1 \cap U_2) \to \psi_1(U_1 \cap U_2)
\]

is a topological map.

When \( \psi_{12} \) is analytic, the manifold \( X=(X, \phi) \) is called n-Dimensional Analytic Manifold. Here the family \( \phi \) is called an analytic structure (or atlas) on X. Every \( x \in U_i \) is determined uniquely by \( \psi_i(x) \). These \( \psi_i \)'s are called local parameters or local variables, [7].

Let \( X=(X, \phi) \) be an analytic manifold and \( W \subset X \) be an open set. Further suppose that \( x_0 \in W \) and \( f \) is a complex valued function on W. If there exists a neighbourhood \( U_{(x_0)} \) of \( x_0 \) so that \( U_{(x_0)} \subset W \cap U_i \) where \( f \circ \psi_i^{-1} \) is holomorphic in \( \psi_i(U_i) \subset B \), then \( f \) is called holomorphic at \( x_0 \). ( \( B \) is an open set in \( \mathbb{C}^n \)) If \( f \) is holomorphic at every point of W then \( f \) is called holomorphic on W. In particular if \( W=X \) then \( f \) is holomorphic on X.

4.3. Theorem. Let X be an analytic manifold of dimension n and \( B(X) \) be a ring of bounded, holomorphic functions (or complex algebra) defined on X. Also suppose that

1) For every \( x \in X \) there exists an \( f \in B(X) \) having a simple zero at \( x \) and no other zeros.

2) For every discrete sequence \( (x_n) \) in X there exists \( f \in B(X) \) so that \( \lim f(x_n) \) does not exist.

Then the necessary and sufficient condition for a maximal ideal in \( B(X) \) to be essential is that it is of the first type.

Proof. First we suppose that \( 1 \in B(X) \) is essential, i.e. \( 1 = \langle f, f = \{gf: g \in B(X)\} \)

\( f \) has a zero. Then \( \inf \{|f(x)|: x \in X\} = 0 \). In this case there exists a sequence \( (x_n) \) in
X so that \( \lim f(x_n) = 0 \). If \( g \in I \) then there exists \( h \in B(X) \) so that \( g = fh \). Since \( h \) is bounded \( \lim g(x_n) = 0 \). Then for every \( g \in B(X) \) \( \lim g(x_n) \) exists. By hypothesis \( (x_n) \) can not be discrete. That is \( x_n \to x \in X \). Therefore the necessary and sufficient condition for \( g \in B(X) \) to be \( g \in I = \{ f \in B(X) : f(x_0) = 0 \} \) then by hypothesis there exists an \( f \in B(X) \) having a simple zero at \( x_0 \) but no other zeros. Now let us think the essential ideal \( < f > \). It is clear that \( f \) is a proper ideal. If \( \phi : B(X) \to \mathbb{C} \), \( \phi(g) = g(x_0) \) is defined then the kernel of \( \phi \) is \( < f > \) and the ideal \( < f > \) is maximal. But as \( I_{x_0} \) is maximal, \( I_{x_0} = < f > \). That is the first type maximal ideal of \( B(X) \) is essential maximal ideal.

REFERENCES


