THE EFFICIENCY ON 2-GENERATORS OF SEMI-DIRECT PRODUCT OF GROUPS

A. S. ÇEVİK

Department of Mathematics, Faculty of Science and Art, Balıkesir University, Balıkesir, TURKEY

(Received Dec. 9, 1998; Revised March. 15, 1999; Accepted March 25, 1999)

ABSTRACT

Let G be a semi-direct product of K by A where K and A are both cyclic groups of order n (n ∈ N) and p (p is a prime), respectively. Then we prove that G has an efficient presentation on the minimal number, that is 2, of generators. After all, as an application of our main result, we give the efficiency of the dihedral group $D_m$ and the metacyclic group of order $2m$ ($m ≥ 4$ and $m$ is even).

1. INTRODUCTION

1.1. EFFICIENCY

Let G be a finitely presented group, and let $P = \langle x; r \rangle$ be a finite presentation for G. Then the Euler characteristic of P is defined by $\chi(P) = 1 - |x| + |r|$ where $| |$ denotes the number of elements in the set. Let

$$\delta(G) = 1 - \mathrm{rk}_2(H_1(G)) + \mathrm{d}(H_2(G)), \quad (1)$$

where $\mathrm{rk}_2(\cdot)$ denotes the $\mathbb{Z}$-rank of the torsion-free part and $\mathrm{d}(\cdot)$ means the minimal number of generators. Then, by [3], [7], [11] for the presentation P, it is always true that $\chi(P) ≥ \delta(G)$. We then define

$$\chi(G) = \min\{\chi(P) : P \text{ is a finite presentation for } G\}$$

We then have the following definition.

Definition 1.1. Let G be a group.

i) A presentation $P_0$ for G is called minimal if
\( \chi(P_0) \leq \chi(P) \),
for all presentations \( P \) of \( G \).

ii) A presentation \( P_0 \) is called efficient if
\( \chi(P_0) = \delta(G) \).

iii) \( G \) is called efficient if
\( \chi(G) = \delta(G) \).

We note that if \( \chi(G) \leq 0 \) then \( G \) must be infinite, and if \( G \) is finite cyclic then \( \chi(G) = 1 \).

Examples of efficient groups are finitely generated abelian groups (Epstein [11]), fundamental groups of closed 3-manifolds [11]; also finite groups with balanced presentations (such finite groups have trivial Schur multiplier [12]). Finite metacyclic groups are efficient. This was shown by Beyl [5] and Wamsley [24]. Infinite metacyclic groups however need not be efficient, a result due to Baik and Pride [3] (see also [1]). In [12] Harlander proved that a finitely presented group embeds into an efficient group. For more references on the efficiency see Baik, Pride [2], Beyl, Rosenberger [6], Campbell, Robertson, Williams [8] (and [9]), Cevik [10], Harlander [13], Johnson, Robertson [15], Kenne [17], Kovacs 18, Swan [23], Wiegold [26].

There is interest not just in finding efficient presentations, but finding presentations which are efficient on the minimal number of generators (see [25]). So here we examine the efficiency of the semi-direct product and then we show that this semi-direct product has an efficient presentation on 2-generators under some certain conditions (see Theorem 1.4 below). Then we give some consequences of this result on the dihedral groups and metacyclic groups.

There is also interest in finding inefficient groups. The following remark is defined to show that a group is inefficient.

**Remark 1.2.** Let \( P \) be a presentation for the group \( G \), and let \( \chi(p) \neq \delta(G) \). Then, by Definition 1.1-(ii), \( P \) is an inefficient presentation. However we can still show that \( P \) is a minimal presentation for the group \( G \) (see [19]). Hence there cannot be an efficient presentation for \( G \). This implies that \( G \) is an inefficient group.

An important remark for our result is the fact that the Schur multiplier \( M(G) \) of a group \( G \) is isomorphic to second integral homology group \( H_2(G) \) of that group.
We should also note that the notation $Z_k$ denotes the cyclic groups of order $k (k \in \mathbb{N})$ in this paper.

1.2. THE DEFINITION OF SEMI-DIRECT PRODUCT

Let $A, K$ be groups, and let $\theta$ be a homomorphism defined by
$$\theta : A \rightarrow \text{Aut}(K), \quad \sigma \rightarrow \theta_\sigma$$
for all $\sigma \in A$. Then the semi-direct product $G = K \rtimes_A A$ of $K$ by $A$ is defined as follows.

The elements of $G$ are all ordered pairs $(\alpha, k)$ ($\alpha \in A, k \in K$) and the multiplication is given by
$$(\alpha, k)(\alpha', k') = (\alpha \alpha', (k \theta_{\alpha'})k').$$

Similar definitions of a semi-direct product can be found in [4] or [22]. The proof of the following lemma can be found in [14, Proposition 10.1, Corollary 10.1].

**Lemma 1.3.** Suppose that $P_K = \langle y; s \rangle$ and $P_A = \langle x; r \rangle$ are presentations for the groups $K$ and $A$ respectively under the maps
$$y \mapsto k_y \quad (y \in y), \quad x \mapsto \alpha_x (x \in X).$$

Then we have a presentation for $G = K \rtimes_A A$
$$P = \langle y, x, s, r, t \rangle$$
where $t = \{yx \lambda_{x^{-1}}^{-1} x^{-1} | y \in y, x \in x \}$ and $\lambda_{yx}$ is a word on $y$ representing the element $(k_y)_{\theta_{\alpha_x}}$ of $K$ ($\alpha \in A$, $x \in K$, $x \in x$, $y \in y$).

1.2. THE MAIN THEOREM

Let $K$ be a cyclic group of order ($n \in \mathbb{N}$) with a presentation $P_K = \langle y; y^n \rangle$, and let $A$ be a cyclic group of order $p$ ($p$ is a prime) with a presentation $P_A = \langle x; x^p \rangle$. Then, by Lemma 1.3 a presentation for $G = K \rtimes_A A$ is given by
$$P = \langle y, x; y^n = 1, x^p = 1, x^{-1}yx = y^r \rangle.$$  \(\text{(2)}\)

where
(i) \((r, n) = 1,\)
(ii) \((r - 1, nt) = t\) with \(t = (r - 1, n),\)
(iii) \(t = p,\)
(iv) \( r^p \equiv 1 \pmod{nt} \),
(v) \( |G| = np \),

for \( r, t \in \mathbb{N} \).

Thus we have the following main theorem.

**Theorem 1.4.** The presentation \( P \), as given in (2), is efficient on 2-generators for the group \( G \).

## 2. PRELIMINARIES AND PROOF OF THE MAIN THEOREM

Though the rest of paper we will assume that \( K \) is a cyclic group of order \( n \) with a presentation \( P_K \), \( A \) is a cyclic group of order \( p \) (\( p \) a prime) with a presentation \( P_A \) and \( G = K \times_\phi A \) with a presentation \( P \) as given in (2) which the conditions (i), (ii), (iii), (iv) and (v) hold.

In 1904, Schur proved that when \( B \) is a finite group then \( H_2(B) \) is a finite group whose elements have order dividing the order of \( B \), and when \( B \) is a (finite or infinite) cyclic group then \( H_2(B) = 1 \) (The details of the proof of these facts can be found in [16]). By using this Schur's results, Park (see [21]) proved the following theorem.

**Theorem 2.1.** Let \( G = K \times_\phi A \). Then \( H_2(G) \) is a cyclic group of order \( p \).

We know that both \( A \) and \( K \) are finite abelian groups, and by conditions (ii)-(iii), since \( (r-1,n) = p \) then \((n,p) \neq 1\). Therefore it is easy to prove the following lemma (see [10] for the details).

**Lemma 2.2.** \( d(A \oplus K) = d(A) + d(K) \).

Now we can prove our main theorem.

In the first part of the proof we will calculate \( \delta(G) \) as given in (1). By condition (v), since \( G \) is a finite group then \( \text{rk}_2 \left( H_1(G) \right) = 0 \), so we will just calculate
\[
\delta(G) = 1 + d(H_2(G)).
\]
But, by Theorem 2.1, since \( H_2(G) \) is a cyclic group of order \( p \) then it is easy to see that \( d(H_2(G)) = 1 \). We then get \( \delta(G) = 2 \).

In the second part of the proof we need to calculate the Euler characteristic of \( P \). In fact, since \( \chi(P) = 1 - (1 + 1) + (1 + 1 + 1) = 2 \) in \( P \), then we get
\[ \delta(G) = \chi(P), \]
so \( P \) is an efficient presentation for \( G \).

In the final part of the proof let us show the efficiency of \( P \) is on 2-generators:
If \( \theta = 1 \) (that is, the trivial homomorphism), then \( G \) is the direct product of the groups \( A \) and \( K \) and so, by Lemma 2.2, \( d(G) = 2 \). If \( \theta \neq 1 \), then \( G \) cannot be abelian and hence \( d(G) \neq 1 \) which gives that \( d(G) = 2 \).

This completes the proof.

**Remark 2.3.** Suppose that \( t = 1 \) in the conditions (ii), (iii) and (iv). Then again in [21] Park proved that \( H_2(G) \) is trivial. Thus we get \( d(H_2(G)) = 0 \). So the above proof implies that \( P \) cannot be an efficient presentation for the group \( G \). But if we show that \( P \) is a minimal presentation then, by Remark 1.2, we can say that \( G \) is an inefficient group.

### 3. APPLICATIONS OF THE MAIN THEOREM

**Example 3.1.** Let us take the dihedral group
\[ D_{10} = \langle y, x; y^{10} = 1, x^2 = 1, x^{-1}yx = y^{-1} \rangle \]
of order 20. Then, by [16] \( H_2(D_{10}) = \mathbb{Z}_2 \) which gives the same result as in Theorem 2.1. Moreover it is easy to see that the five conditions in Section 1.3 hold. Hence, by Theorem 1.4, we say that the above presentation of the group \( D_{10} \) is efficient on 2-generators.

Therefore, as an application of Theorem 1.4, we can generalize this example as follows.

**Corollary 3.2.** The dihedral group
\[ D_m = \langle y, x; y^m = 1, x^2 = 1, x^{-1}yx = y^{-1} \rangle \]
of order \( 2m \) (\( m \geq 4 \) and is even) is efficient.

**Proof.** It is easy to see that the conditions (i), (ii), (iii), (iv) and (v) in Section 1.3 hold. Also, as a consequence of Theorem 2.1, in [21] Park proved that \( H_2(D_m) \) is a cyclic group of order 2. Then, by Theorem 1.4, \( D_m \) is efficient on 2-generators.

We should note that for \( m \geq 3 \) and \( m \) is odd, the above proof cannot cover the efficiency of \( D_m \).
Remark 3.3. In Example 3.1, one can show that the dihedral group $D_{10}$ is isomorphic to the metacyclic group, say $B$, of order 20. Thus, by Theorem 1.4, we say that $B$ has an efficient presentation on 2-generators. In fact, we can extend the isomorphism between these two groups to their general form under some conditions.

Therefore, as a consequence of the Theorem 1.4, we have the following well-known result (see also [5] or [24]).

Corollary 3.4. Let $B$ be a metacyclic group of order $2m$ ($m \geq 4$ and $m$ is even) with a presentation

$$P_B = \langle a, b; a^m = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

Then $P_B$ is an efficient presentation on 2-generators for the group $B$.

Proof. A similar proof can be applied as the proof of Corollary 3.2.

ACKNOWLEDGEMENT. I would like to express my deepest thanks to Prof. Dr. Turgut Başkan for his helpful and useful comments.

REFERENCES


