ON THE RING OF THE HOLOMORPHIC FUNCTIONS OVER THE ALGEBRA

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(Received Jan. 12, 1999; Accepted Nov. 3, 1999)

ABSTRACT

We prove that there exists a diffeomorphism between any subsets $G^1_m$ and $G^2_m$ of an algebra $A$ of which nonzero elements are regular if there is an $A$-isomorphism between the rings $H(G^1_m)$ and $H(G^2_m)$.

1. INTRODUCTION

Working on conformal equivalence by means of the ring of analytic functions began in year 1940 [2].

Let $G_1$ and $G_2$ be two domains in the complex plane, and let $A(G_1)$ and $A(G_2)$ be the rings of analytic functions on them. If there exists a C-isomorphism between $A(G_1)$ and $A(G_2)$, then $G_1$ and $G_2$ are conformally equivalent, where $C$ is the set of complex numbers [1]. The problem was generalized to open Riemann surfaces $G_1$ and $G_2$ [5]. It was shown that two domains $G_1$ and $G_2$ in the complex plane were conformally equivalent if the rings $B(G_1)$ and $B(G_2)$ of all bounded analytic functions defined on them were algebraically C-isomorphic [3]. When we discuss the rings $B(G_i)$ ($i = 1, 2$), it is always assumed that $G_i$ is bounded and has the following property: for any $z \in \partial G_i$, boundary of $G_i$, there exists a function $f \in B(G_i)$ for which $z$ is an unremovable singularity. It is proved that if there is a C-isomorphism between $A(G_1)$ and $A(G_2)$, then the sets $G_1$ and $G_2$ are conformally equivalent [7].
Now our aim is to investigate the above problem for the algebra \( A \) with finite dimensional.

2. THE HOLOMORPHIC FUNCTIONS OVER AN ALGEBRA

Let \( A \) be an associative commutative unital algebra of finite dimension \( m \) over the field \( R \) of real numbers. We have
\[
e_{\alpha}e_{\beta} = C_{\alpha\beta}^{\gamma}e_{\gamma}, \quad (\alpha, \beta, \gamma = 1, \ldots, m)
\]
such that the set \( \{e_1, e_2, \ldots, e_m\} \) is a basis of the algebra \( A \), where \( C_{\alpha\beta}^{\gamma} \) is a new notation for \( \sum_{\gamma=1}^{m} C_{\alpha\beta}^{\gamma}e_{\gamma} \), i.e., \( C_{\alpha\beta}^{\gamma}e_{\gamma} = \sum_{\gamma=1}^{m} C_{\alpha\beta}^{\gamma}e_{\gamma} \) called the Einstein symbol. The coefficients \( C_{\alpha\beta}^{\gamma} \) are called the structure constants of the algebra \( A \). The structure constants are the components of the tensor field of type (1,2).

By using structure constants, in order to show that \( A \) is commutative, associative and unital algebra, we have
\[
C_{\alpha\beta}^{\gamma} = C_{\beta\alpha}^{\gamma}
\]
\[
C_{\alpha\beta}^{\gamma}C_{\beta\rho}^{\delta} = C_{\alpha\rho}^{\delta}C_{\beta\beta}^{\gamma}
\]
\[
C_{\alpha\beta}^{\gamma}e_{\beta} = C_{\beta\alpha}^{\gamma}e_{\beta} = \delta_{\alpha}^{\gamma}
\]
respectively. Where \( e_{\beta} \) is component of \( 1 \) which is the unit of \( A \) such that \( 1 = \epsilon_{\beta}^{\beta}e_{\beta} \) and \( \delta_{\alpha}^{\gamma} \) is Kronecker's symbol. In this paper, we assume that \( A \) is an associative commutative unital algebra.

Let \( X = x^{\alpha}e_{\alpha}, \quad \alpha = 1, \ldots, m \), be a variable in the algebra \( A \), where \( e_{\alpha} \) and \( x^{\alpha} \) denote the basis units of \( A \) and real variables, respectively. Then the function
\[
F = f^{\alpha}e_{\alpha}
\]
defined over the algebra \( A \) is a function in \( X \), where \( f^{\alpha} = f^{\alpha}(x^1, \ldots, x^m) \) are real functions in all \( x^{\alpha} \). We have \( F = F(X) \). Let us define the differential in \( A \) by
\[
dX = dx^{\alpha}e_{\alpha} \quad \text{ve} \quad dF = df^{\alpha}e_{\alpha}.
\]

If the differential \( dF \) can be represented in the form \( dF = F(X) dX \), then \( F = F(X) \) is said to be \( A \)-holomorphic (\( A \)-differentiable), where \( F(X) \) represents the derivative of \( F(X) \) [4, 6, 8].
**Theorem 2.1.** The function $F = F(X)$ is A-holomorphic if and only if

$$C_\alpha D = DC_\alpha$$  \hspace{1cm} (1)$$

where $C_\alpha = (C^\gamma_{\alpha \beta})$ are structure constants matrix and $D = \left( \frac{\partial f^\alpha}{\partial x^\beta} \right)$ is real Jacobian matrix such that $\gamma$ and $\beta$ represent row and column, respectively [4].

The equality (1) are called Schaffers conditions[4]. In particular, if $A = C$ is the complex number algebra $(m = 2)$, the Schaffers conditions coincide with the Cauchy-Riemann conditions: Let us consider the algebra $C = \mathbb{R}(i)$, $i^2 = -1$. The dimension of the algebra $C$ is 2. The basis of the algebra $C$ is the set $\{e_1, e_2\}$ such that $e_1 = 1$, $e_2 = i$. If the equality

$$e_i e_j = C^i_{jj} e_1 + C^i_{ij} e_2$$

is considered, we have the structure constants matrices

$$C_1 = \begin{pmatrix} C^1_{11} & C^1_{12} \\ C^1_{11} & C^1_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} C^2_{21} & C^2_{22} \\ C^2_{11} & C^2_{22} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

The Jacobian matrix of the function $f(z) = u(x, y) + iv(x, y)$ is that

$$D = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$ 

Using the Schaffers conditions $C_\alpha D = DC_\alpha$, $\alpha = 1, 2$ we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ 

or shortly

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

or shortly

$$\begin{pmatrix} -v_x & -v_y \\ u_x & u_y \end{pmatrix} = \begin{pmatrix} u_y & -u_x \\ v_y & -v_x \end{pmatrix}.$$
Therefore from the equalities $C_1D = DC_1$ and $C_2D = DC_2$, we obtained
\[ u_x = v_y, \quad u_y = -v_x \]
known as the Cauchy-Riemann conditions.

Note that, generally, A-holomorphic functions and analytic functions are different in algebra A [8].

3. DIFFEOMORPHISM BETWEEN $G_m^1$ AND $G_m^2$

Let $G_m^i$, $(i = 1,2)$ be subsets of the algebra A. If there exists a bijective function $f: G_m^1 \rightarrow G_m^2$ such that the functions $f$ and $f^{-1}$ are A-differentiable, then the function $f$ is called an A-diffeomorphism. If $f$ is an A-diffeomorphism, the determinant of the Jacobian matrix $D_f$ of the function $f$ is not zero. That is, $|D_f| > 0$ or $|D_f| < 0$, where $|D_f|$ is determinant of the Jacobian matrix $D_f$. Note that, if we consider the set of the complex numbers C, then a diffeomorphism, generally, is not a conformal mapping. Let $F: G_m^1 \rightarrow A$ be a A-holomorphic mapping and $\varphi: G_m^1 \rightarrow G_m^2$ be A-diffeomorphism. Since
\[ C_\alpha D_{f \circ \varphi^{-1}} = D_{f \circ \varphi^{-1}} C_\alpha, \]

$F \circ \varphi^{-1}: G_m^2 \rightarrow A$ is a A-holomorphic mapping, where $D_{f \circ \varphi^{-1}}$ is the Jacobian matrix of $F \circ \varphi^{-1}$.

On the other hand the sets

\[ H(G_m^i) = \{ F: G_m^i \rightarrow A : F \text{ is a } A \text{-holomorphic function} \} \quad (i = 1,2) \]

become a ring under the operations

$(F + G)(X) = F(X) + G(X)$ and $(FG)(X) = F(X)G(X)$.

Theorem 3.1. If $\varphi: G_m^1 \rightarrow G_m^2$ is a diffeomorphism, then $\Phi: H(G_m^1) \rightarrow H(G_m^2)$. $\Phi(F) = F \circ \varphi^{-1}$ is an A-isomorphism, i.e., the isomorphism $\Phi$ satisfies $\Phi(\mathcal{A}) = \mathcal{A}$ for every $\mathcal{A} \in A$.

Proof. It is easily shown that $\Phi$ is bijective and $\Phi(F + G) = \Phi(F) + \Phi(G)$ and $\Phi(FG) = \Phi(F)\Phi(G)$. Thus $\Phi$ is an isomorphism. On the other hand $\Phi(\mathcal{A}) = \mathcal{A}$ for all $\mathcal{A} \in A$. Thus $\Phi$ is an A-isomorphism.
Definition 3.2. Let $\alpha$ be a nonzero element of the algebra $A$. If there exists $\beta \in A$ such that $\alpha \beta = 1$, then $\alpha \in A$ is said to be a regular element.

For each $\alpha \in G_m^i$ ($i = 1, 2$), we consider the set

$$M(\alpha) = \{ F \in H(G_m^i) : F(\alpha) = 0 \}.$$

Lemma 3.3. $M(\alpha)$ is a principal ideal of $H(G_m^i)$ ($i = 1, 2$) generated by the function $X - \alpha$.

Proof. The proof is clear.

Now, suppose that nonzero elements of the algebra $A$ are regular. We can write the following lemma such that $G_m^i$ ($i = 1, 2$) is a subset of $A$.

Lemma 3.4. $M(\alpha)$ is a maximal ideal of $H(G_m^i)$.

Proof. For instance, suppose that $M(\alpha)$ is not a maximal ideal. In that case there exists an ideal $I$ which contains $M(\alpha)$. There exists a function $G$ such that $G \in I$ and $G \notin M(\alpha)$. Thus, $G(\alpha) \neq 0$. If $H(X) = G(X) - G(\alpha)$, then $H \in M(\alpha) \subset I$. Hence, we have $G(\alpha) = G(X) - H(X)$. Since $G(\alpha) \in A$ is a regular element, we have $I = H(G_m^i)$. Hence, the assertion holds.

Definition 3.5. $M(\alpha)$ is called a fixed maximal ideal of $H(G_m^i)$. All other maximal ideals of $H(G_m^i)$ are called free maximal ideals.

Theorem 3.6. If $\Phi : H(G_m^1) \rightarrow H(G_m^2)$ is an $A$-isomorphism, then there exists an $A$-diffeomorphism between $G_m^1$ and $G_m^2$.

Proof. Let $\Phi : H(G_m^1) \rightarrow H(G_m^2)$ be an $A$-isomorphism. Then, to every fixed maximal ideal $M(\alpha)$ of $H(G_m^1)$ corresponds to a fixed maximal ideal $M'(\alpha')$ of $H(G_m^2)$. If we put $\alpha' = \phi(\alpha)$, then $\phi : G_m^1 \rightarrow G_m^2$ is a bijective mapping. In order to prove that $\phi : G_m^1 \rightarrow G_m^2$ is an $A$-diffeomorphism, let us put $G_v(X) = X$ on $G_m^2$. Then $G_v \in H(G_m^2)$. Since $\Phi$ is an $A$-isomorphism, there exists $F_v \in H(G_m^1)$ such that $\Phi^{-1}(G_v) = F_v$. It is then easy to see that, for any $\alpha \in G_m^1$, $F_v(X) - F_v(\alpha) \in M(\alpha)$ and $\Phi(F_v(X) - F_v(\alpha)) \in M'(\alpha')$ hence
\[ G(X) - F_0(\alpha) = X - F_0(\alpha) \in M'(\alpha') = M' (\varphi(\alpha)) . \]

This shows that \( \varphi \in H(G_m^2) \), i.e. \( \varphi \) is a \( A \)-holomorphic function. Similarly, we can also show that \( \varphi^{-1} \) is an \( A \)-holomorphic function. Hence, \( \varphi \) is an \( A \)-diffeomorphism.

ACKNOWLEDGEMENT.

Author thanks Prof. Dr. A. A. Salimow for helping in Russian references.

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