LORENTZIAN CIRCLE AS A LIE GROUP AND A $C^\infty$ ACTION ON LORENTZ SPACES OF TWO DIMENSION

H. GÜNDOĞAN* and B. KARAKAŞ**

*Department of Mathematics, University of Kirikkale, Kirikkale-Turkey
**Department of Mathematics, University of Yüzüncü Yıl, Van-Turkey

ABSTRACT

We define a product on Lorentz circle that is similar to Lorentzian inner product. This product is expanded to Lorentz space of two dimension. So we have an $C^\infty$ action of Lorentz circle on Lorentz space of two dimension. It is noted that this action provides some isometrics of $L^2$.

1. INTRODUCTION

A Lie group $G$ is a group which has the structure of a differentiable manifold and for which the group function

$O : G \times G \rightarrow G$

defined by $O(g_1, g_2) = g_1 g_2$ is differentiable. Given an element $a$ of a Lie group $G$, the function $L : G \rightarrow G$ defined $g \rightarrow ag$ is called left translation. A Lie group $G$ is said to act on a differentiable manifold $M$ as Lie transformation group if we are given a global surjection

$\phi : G \times M \rightarrow M$

which is differentiable such that if $g, h \in G$ and $m \in M$

$\phi(g, \phi(h, m)) = \phi(gh, m)$

$G$ is said act transitively on $M$ if, given any two points $m_1, m_2 \in M$ there is an element $g \in G$ such that $m_2 = gm_1$. 
By a transformation of a manifold $M$, we mean a differomorphism of $M$ onto itself. A group $G$ is said to act on $M$ as a transformation group. If there is a global function

$$\phi : G \times M \to M$$

such that

i) the function $\phi_g$, defined for any given $g \in G$ by $m \to \phi(g, m)$ is a transformation of $M$,

ii) if $g, h \in G$, $\phi_g \circ \phi_h = \phi_{gh}$.

Suppose that $e$ is the unit element of $G$, then $\phi_e$ is the identity element on $M$, for if $m \in M$ and $m' = (\phi_e - 1)_m$. The group is said to act effectively on $M$ if $e$ is the only element of $G$ such that $\phi_g m = m$ for all $m \in M$. It is said to act freely on $M$ if $e$ is the only element of $G$ such that $\phi_g m = m$ for some $M$.

A transformation group $G$ acting on a manifold $M$ sets up an equivalence relation on $M$. The equivalence class containing a point $m$ is the range of the function $\phi_m : G \to M$ and we call it the orbit of $m$ [1].

We will denote Lorenzian circle on $L^2$ by $L^1$. In this study, it was shown that $L^1$ is a Lie group with a binary operation defined on $L^1$ as hypercylindrical product defined by [3] A $C^\infty$- action on $L^2$ of this Lie group is defined and some properties of it were given. Finally, with the help of this $C^\infty$-action some isometries of $L^2$ were obtained.

### 1.1 Lorentz Manifolds, Lorentz Vector Spaces and Lorentz Circle

A metric tensor $g$ on a differentiable manifold $M$ is a symmetric nondegenerate $(0,2)$ tensor field on $M$ of constant index. A semi-Riemannian manifold is a differentiable manifold $M$ furnished with a metric tensor $g$. Thus a semi-Riemannian manifold is an ordered pair $(M, g)$.

The common value $v$ of index $g_p$ on a semi-Riemannian manifold $M$ is called the index of $M$ satisfying $0 \leq v \leq n = \dim M$. If $v = 0$, $M$ is a Riemannian manifold since each $g_p$ is an (positive definite ) inner product on $T_p(M)$ If $v = 1$ and $n \leq 2$, $M$ is Lorentz manifold.
In this study, \( v \) will take the values 1 and 2, and \( L^2 \) will denote a Lorentz manifold of two dimension. So, the metric tensor is defined by

\[
< X, Y >_L = x_1 y_1 - x_2 y_2
\]

or

\[
< X, Y >_L = < X, Y_s >_E
\]

where \( Y_s \) is symmetry of \( Y \) according to the x-axes and \( X = (x_1, x_2), Y = (y_1, y_2) \) [4].

## 2 LIE GROUP STRUCTURE OF \( L_1^1 \)

We define a binary operation on \( L_1^1 \) by

\[
O : L_1^1 \times L_1^1 \rightarrow L_1^1
\]

\[
O (X, Y) = (\langle X, Y \rangle_E, \langle X, Y_s \rangle_E)
\]

where \( \langle , \rangle_E \) is Euclidean inner product and \( Y_s \) is symmetry of \( Y \in \mathbb{R}^2 \) according to the straight line \( y = x \). We have, the following

**Theorem 1.** The system \((L_1^1, O)\) is a commutative group.

**Proof.** For all \( X, Y \in L_1^1 \),

1. \( O (X, Y) \in L_1^1 \),

2. \( O (X, Y) = (x_1 y_1 + x_2 y_2, x_1 y_2 + x_2 y_1) \)

\[
= (x_1 y_1 + x_2 y_2, y_1 x_2 + y_2 x_1)
\]

\[
= O (Y, X)
\]

3. \( e = (1, 0) \) is the identity element

4. The inverse element of \( X = (x_1, x_2) \) is \( (x_1 - x_2) \).

By Theorem 1, \( L_1^1 \) becomes a Lie group since \( L_1^1 \subset \mathbb{R}^2 \) is a differentiable submanifold, and the symmetry function and the inner product are differentiable functions.

For \( r \in \mathbb{R}^+ \), we define the set \( L_r^1 \) as

\[
L_r^1 = \left\{ (x, y) \in L^2 \mid x^2 - y^2 = r^2 \right\}
\]

and action \( \theta : L_1^1 \times L_r^1 \rightarrow L_r^1, \theta((x_1, x_2), (y_1, y_2)) = (x_1 y_1 + x_2 y_2, x_1 y_2 + x_2 y_1) \).
Theorem 2. $L^1_r$ acts transitively on $L^1_r$.

Proof. For $p, q \in L^1_r$ we can define an element $X = (x_1, x_2)$ where
\[
x_1 = \frac{<p, q>}{r^2} \quad \text{and} \quad x_2 = \frac{\langle p, q_s \rangle}{r^2}.
\]
Then, it is clear that $X \in L^1_r$ and
\[
\theta(X, Q) = P
\]
This completes the proof.

For $X \in L^1_r, (\theta)_X$ the orbit of $X$ under the action $\theta$ is $L^1_r$.

Theorem 3. $L^1_r$ acts effectively on $L^1_r$.

Proof. We have to show that for all $m \in L^1_r$ the equation $\theta(g, m) = m$ is satisfied only for $g = e$. In fact
\[
\theta(g, m) = m \Rightarrow g_1 m_1 + g_2 m_2 = m_1
\]
\[
g_1 m_2 + g_2 m_1 = m_2
\]
\[
\det \begin{bmatrix} m_1 & m_2 \\ m_2 & m_1 \end{bmatrix} = 1
\]
\[
\Rightarrow g_1 = \frac{\begin{bmatrix} m_1 & m_2 \\ m_2 & m_1 \end{bmatrix}}{r^2} = 1
\]
\[
\det \begin{bmatrix} m_1 & m_1 \\ m_2 & m_2 \end{bmatrix} = 0
\]
so $g = e$.

3. THE SET $\tilde{L}^1_r$

We define the set $\tilde{L}^1_r$ as
\[
\tilde{L}^1_r = \{ (x, y) \mid y^2 - x^2 = r^2 \}
\]
so we can define an action
\[
\bar{O} : \tilde{L}^1_r \times \tilde{L}^1_r \longrightarrow \tilde{L}^1_r
\]
\[
\bar{O}(X, Y) = \langle (X, Y)_E, (X, Y_s)_E \rangle
\]
In this case, we evidently have;
1. $(\tilde{L}^1_r, \bar{O})$ is a commutative group
2. $\tilde{L}^1_r$ is a Lie group
3. \( \mathcal{L}_1^1 \) acts on \( \mathcal{L}_r^1 \) as a Lie transformation group with the function \( \tilde{\theta} \) defined by
\[
\tilde{\theta}(g, X) = (\langle g, X \rangle_E, \langle g, X_S \rangle_E)
\]
4. \( \mathcal{L}_1^1 \) acts transitively on \( \mathcal{L}_r^1 \)
5. \( \mathcal{L}_1^1 \) acts effectively on \( \mathcal{L}_r^1 \)
6. \( (\tilde{\theta})_X = \tilde{\mathcal{L}}_r^1 \), where \( X \in \tilde{\mathcal{L}}_r^1 \)

### 3.1. AN ACTION ON \( L^2 \)

For all \( X = (x_1, x_2) \in L^2 \), we have
\[
x_1^2 - x_2^2 = r^2 \quad \text{or} \quad x_2^2 - x_1^2 = r^2
\]
where \( r \in \mathbb{R}^+ \cup \{0\} \). So, we write
\[
(\bigcup_{r} \mathcal{L}_r^1) \cup (\bigcup_{r} \tilde{\mathcal{L}}_r^1) \supseteq L^2.
\]
Then, we conclude

**Theorem 4.** \( L_1^1 \) acts on \( L^2 \) as a Lie transformation group with the function \( \theta' \) defined by
\[
\theta'(X, Y) = \begin{cases} 
\theta(X, Y), & \text{if } y_1 \geq y_2 \\
\tilde{\theta}(X, Y), & \text{if } y_1 < y_2
\end{cases}
\]

For all \( p \in L^2 \), the orbit of \( p = (p_1, p_2) \) under \( \theta' \) is
\[
(L_1^1)_{(p)} = \begin{cases} 
L_r^1, & p_1 \geq p_2 \\
\tilde{\mathcal{L}}_r^1, & p_1 < p_2
\end{cases}
\]

Also, for all \( g \in L^2 \) the mappings \( \theta' : L^2 \longrightarrow L^2 \) defined by
\[
\theta'_g(X) = \theta'(g, X)
\]
are diffeomorphisms.

**Theorem 5.** The mappings \( \theta'_g \) are isometrics of \( L^2 \).

**Proof.** Let \( X = (x_1, x_2), \ Y = (y_1, y_2) \in L^2 \) and \( g = (g_1, g_2) \in L_1^1 \). Thus
\[
d_L(\theta'_g(X), \theta'_g(Y))^2 = d_L[(\langle g, X \rangle_E, \langle g, X_S \rangle_E), (\langle g, Y \rangle, \langle g, Y_S \rangle)]^2
\]
\[(\langle g, X \rangle - \langle g, Y \rangle)^2 - \langle \langle g, X_S \rangle, \langle g, Y_S \rangle \rangle^2 \]
\[= \langle g, X - Y \rangle^2 - \langle g, X_S - Y_S \rangle^2 \]
\[= (g_1(x_1 - y_1) + g_2(x_2 - y_2))^2 - (g_1(x_2 - y_2) + g_2(x_1 - y_1))^2 \]
\[= g_1^2(x_1 - y_1)^2 + g_2^2(x_2 - y_2)^2 + 2g_1g_2(x_1 - y_1)(x_2 - y_2) \]
\[- g_1^2(x_2 - y_2)^2 - g_2^2(x_1 - y_1)^2 - 2g_1g_2(x_2 - y_2)(x_1 - y_1) \]
\[= (g_1^2 - g_2^2)(x_1 - y_1)^2 - (g_1^2 - g_2^2)(x_2 - y_2)^2 \]
\[= (x_1 - y_1)^2 - (x_2 - y_2)^2 \]
\[= d_1(X, Y)^2 \]

where \(x_1 < x_2, y_1 < y_2\). All other possibilities, which are \(x_1 \geq x_2, y_1 < y_2\) or \(x_1 \geq x_2, y_1 \geq y_2\) or \(x_1 < x_2, y_1 \geq y_2\) can be verified as above.

REFERENCES