THE CONJUGATE OF A HYPERSURFACE

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ABSTRACT

In this study, the idea of the conjugate of a surface in $E^3$ given by TH. Hasantis and D. Koutoufiotis [3] has been generalized for a hypersurface in $E^{n+1}$. A necessary and sufficient condition for having the conjugate of a hypersurface has been given. Gauss and mean curvatures of the conjugate hypersurface have also been calculated.

1. INTRODUCTION

Let $M$ be a smooth immersed regular hypersurface in $E^{n+1}$, which is connected and oriented. Let us choose $O \in E^{n+1}$ as an origin. We denote by $x$ the position vector of a point in $M$, and set $|x| = r$ for the corresponding distance function. Let $N$ be the unit normal vector field of $M$. The support function $f$ of $M$ with respect to $O$ is defined as $f = \langle x, N \rangle$, which is also differentiable, where $\langle \ , \ \rangle$ is the inner product on $E^{n+1}$. Let $(u^1, \ldots, u^n)$ be a local coordinate system on $M$. We denote the components of the first, second and third fundamental forms, respectively, by $g_{ij} = \langle x_i, x_j \rangle$, $b_{ij} = \langle x_i, N_j \rangle$ and $n_{ij} = \langle N_i, N_j \rangle$, where $x_i = \frac{\partial x}{\partial u^i}$ and $N_i = \frac{\partial N}{\partial u^i}$.

Let $\nabla$ be the standard connection of $E^{n+1}$, $\nabla$ be the induced connection on $M$. The equations of Gauss and Weingarten are, respectively,

$$\nabla_x Y = \nabla_x Y + (AX, Y) N,$$  

(1.1)

and

$$\nabla_x N = - AX$$  

(1.2)

where $X$ and $Y$ are vector fields tangent to $M$ and $A$ is the Weingarten mapping of $M$. The eigenvalues of $A$ are the principal curvatures.
The Gauss curvature is $K = k_1 k_2 \ldots k_n$ and the mean curvatures is $H = \frac{1}{n} \sum_{i=1}^{n} k_i$.

Suppose now that there exist a point $O$ with the property that it lies on no tangent hyperplane of $M$. If we choose such a point as origin, the corresponding support function clearly never vanishes. So, either $f > 0$ or $f < 0$. We can always choose an orientation of $M$ which makes $f > 0$. Thus, $M$ is obviously star-shaped.

We decompose the position vector $x$ of a point of $M$ into two parts: a component normal to $M$, and a component tangent to $M$ such that

$$x = x_T - fN.$$ (1.3)

Let $X$ be a tangent vector of $M$. Since $\nabla_X x = X$,

$$X = \nabla_X x = \nabla_X (x_T - fN) = \nabla_X x_T - (Xf)N - f \nabla_X N$$

or

$$X = \nabla_X x_T + \langle AX, x_T \rangle N -(Xf)N + f AX.$$  

Taking the tangential component of this equation, we obtain

$$\nabla_X x_T = (I - fA)X,$$ (1.4)

where $I$ is the identity transformation, and taking the normal component we obtain

$$\langle AX, x_T \rangle N = (Xf)N$$

or

$$\langle X, Ax_T \rangle = \langle X, \text{grad } f \rangle.$$  

So that

$$Ax_T = \text{grad } f.$$ (1.5)

Furthermore, since
X(r^2) = X((x, x))
    = 2(\nabla_x x, x)
    = 2(X, x_{\tau}),

or

X(r^2) = 2rX(r)
    = 2r(X, \text{grad } r),

then

\text{grad } r = \frac{X_{\tau}}{r}, \quad (1.6)

2. THE CHARACTERISTIC MAPPING OF A HYPERSURFACE

Let M be oriented hypersurface and S^n be the unit hypersphere centered at O. We define the smooth mapping \( \zeta : M \rightarrow S^n \) by

\[ \zeta(x) = \frac{x + 2fN}{r} \]

Further, we define the mapping \( \eta : M \rightarrow S^n \) by

\[ \eta(x) = e = \frac{x}{r} \]

that is, \( \eta \) is a diffeomorphism of M onto the open subset \( A = \eta(M) \) of \( S^n \). Then we can define the characteristic mapping \( \tau : A \rightarrow S^n \) of M, where \( \tau = \zeta \circ \eta^{-1} \) by. Obviously, the position vector \( e \) of a point in \( A \) with respect to O can be written as

\[ \tau(e) = e + \frac{2fN}{r} \quad (2.1) \]

Let \( (u^1, \ldots, u^n) \) be a local coordinate system of A, so we write \( e_i = \frac{\partial e}{\partial u^i} \)
and \( \tau_i = \frac{\partial \tau}{\partial u^i} \). From (2.1)

\[ 1 - \langle \tau(e), e \rangle = \frac{2f^2}{r^2} \quad (2.2) \]

Then, \( \tau \) can have no fixed points. Instead of \( \tau(e) \), we write simply \( \tau \) and using \( e = \frac{x}{r} \), after a brief calculation we obtain

\[ \langle \tau, e \rangle = \frac{2f^2}{r^2} \frac{\partial}{\partial u^i} (\log r), \quad 1 \leq i \leq n \quad (2.3) \]
From (2.2) and (2.3), we find the first-order system of differential equations
\[
\frac{\partial}{\partial u^i} \left( \log r \right) = \frac{\langle \tau, e_i \rangle}{1 - \langle \tau, e \rangle}, \quad 1 \leq i \leq n.
\] (2.4)

The integrability conditions for this system can be written as
\[
\frac{\partial}{\partial u^i} \left[ \frac{\langle \tau, e_j \rangle}{1 - \langle \tau, e \rangle} \right] = \frac{\partial}{\partial u^j} \left[ \frac{\langle \tau, e_i \rangle}{1 - \langle \tau, e \rangle} \right], \quad 1 \leq i, j \leq n,
\]
or
\[
\langle \tau, e_j \rangle - \langle \tau, e_i \rangle = \frac{\langle \tau, e_j \rangle \langle \tau, e_i \rangle - \langle \tau, e_i \rangle \langle \tau, e_j \rangle}{1 - \langle \tau, e \rangle}.
\] (2.5)

The length of the position vector \( r \) of \( \mathcal{M} \) satisfies the differential equations system (2.4). If a given mapping \( \tau : A \to \mathbb{S}^n \) without fixed points is the characteristic mapping of a hypersurface, then the corresponding hypersurface \( \mathcal{M} \) is given by its position vector \( x = r e \).

### 3. THE CONJUGATE OF A HYPERSURFACE

Let \( \mathbb{S}^n \) be unit hypersphere centered at \( O \) and \( e \) be the position vector of \( \mathbb{S}^n \). The mapping \( \alpha : \mathbb{S}^n \to \mathbb{S}^n, \alpha(e) = -e \), is called as an antipodal mapping. If a given the characteristic mapping \( \tau \) of a hypersurface \( \mathcal{M} \), we set \( \tilde{\tau} = \alpha \circ \tau \).

**Definition 3.1.** Let \( \tau \) be the characteristic mapping of a hypersurface \( \mathcal{M} \) in \( \mathbb{E}^{n+1} \). If \( \tilde{\tau} \) also the characteristic mapping of some hypersurface \( \overline{\mathcal{M}} \), then \( \overline{\mathcal{M}} \) is called the conjugate hypersurface of \( \mathcal{M} \).

If \( \tilde{\tau} \) is the characteristic mapping of an \( \overline{\mathcal{M}} \), then \( \tilde{\tau} \) has no fixed points.

**Theorem 3.2.** The hypersurface \( \mathcal{M} \) has the conjugate \( \overline{\mathcal{M}} \) if and only if grad \( r \neq 0 \) and the vector field grad \( r \), grad \( f \) on \( \mathcal{M} \) are linear depended at every point.

**Proof.** Suppose \( \mathcal{M} \) has the conjugate \( \overline{\mathcal{M}} \). Then \( \tilde{\tau} \) has no fixed points, that is, \( \tau(e) \neq -e \) for every \( e \) in the domain of \( \tau \). This means that \( x \) is never perpendicular to \( \mathcal{M} \), and since grad \( r = \sum_{j=1}^{n} r_j \frac{\partial}{\partial u^i}, \quad r_i = \frac{\partial r}{\partial u^i} = \frac{\langle x, x \rangle}{r}, \)
grad \( r \neq 0 \). Considering the integrability condition \((2.5)\) for \( \tau \) and \( \overline{\tau} \), we obtain

\[
\langle \tau, e_j \rangle = \langle \tau, e_j \rangle .
\]

(3.1)

From \((3.1)\), we compute

\[
\langle \tau, e_j \rangle - \langle \tau, e_i \rangle = \frac{4f}{r^3} \left( r_i f_j - f_i r_j \right) = 0 ,
\]

or

\[
f_{i j} = f_{j i} .
\]

Thus, the vector fields \( \text{grad} \, r \), \( \text{grad} \, f \) are linear depended.

Conversely \( \text{grad} \, r \neq 0 \) and the vector fields \( \text{grad} \, r \), \( \text{grad} \, f \) are linear depended. Since \( \text{grad} \, r \neq 0 \) the mapping \( \overline{\tau} = \alpha \circ \tau \) has no fixed points. Since the \( \text{grad} \, r \) and \( \text{grad} \, f \) are linear depended, the equality \((3.1)\) holds. Hence, the \( \overline{\tau} \) satisfies the integrability condition \((2.5)\), that is \( M \) has the conjugate \( \overline{M} \).

Theorem 3.2 holds for a hypersurface \( M \). From \((1.5)\) and \((1.6)\)

\[Ax_{\tau} = \text{grad} \, f = c \text{grad} \, r = \frac{c}{r} x_{\tau} , \quad c \neq 0 , \quad c \in \mathbb{R},\]

this means the vector \( x_{\tau} \) is the eigen vector of \( A \). Thus, \( M \) has conjugate hypersurface if and only if the tangential component \( x_{\tau} \) of the position vector \( x \) of \( M \) is the eigen vector of \( A \). Setting \( X = x_{\tau} \) in \((1.4)\), we obtain

\[
\nabla_{x_{\tau}} x_{\tau} = (1 - f k) x_{\tau} ,
\]

where \( k_1 \) is the principal curvature the corresponding to \( x_{\tau} \).

Since the position vector of \( M \) can be written as \( x = r e \), we write \( \overline{x} = \hat{r} e \), where \( \overline{x} \) is the position vector of \( \overline{M} \). Moreover \( \frac{\overline{x}}{r} = \overline{X} \) and \( \overline{\tau}(e) = -\tau(e) \). So,

\[
\frac{x}{r} + \frac{2 f N}{r} = \frac{x}{r} - \frac{2 f \overline{N}}{r} .
\]

This relation tells us that \( \overline{N} \) is the hyperplane spanned by \( x \) and \( N \). We compute \( \langle \overline{N}, N \rangle = 0 \), hence \( \overline{N} \) is parallel to \( x_{\tau} \). For the position vector of \( \overline{M} \), we write
\[ \bar{x} = \frac{\bar{r}}{r} x = \frac{\bar{r}}{r} (x_T - f N), \]
or
\[ \bar{x} = \bar{x}_T - \bar{f} N. \]

From this we obtain \( \bar{f} = -\frac{\bar{x}_T}{N} = -\frac{\bar{r}}{r} x_T, N \). Since \( N \) is parallel to \( x_T \), we choose \( N = \frac{-x_T}{|x_T|} \), which makes \( \bar{f} \) positive and
\[ \bar{f} = \frac{\bar{r}}{r} |x_T|. \]

**Theorem 3.3.** The natural mapping from \( M \) to \( \bar{M} \) preserves principal directions. Moreover, the corresponding principal curvatures at corresponding points are related by
\[ k_i = \frac{\bar{f}}{r^2} k_i, \quad \bar{k}_i = \frac{1 - f k_i}{\bar{r}} \frac{k_i}{r^2}, \quad 2 \leq i \leq n, \]

where \( k_i \) is the principal curvature in the direction \( x_i \).

**Proof.** Let \( (u^1, u^2, \ldots, u^n) \) be the local coordinate system in the neighbourhood of a point of \( M \) which is not an umbilic. Let the parameter curves of \( M \) be the curvature lines. Since, \( M \) has the conjugate \( \bar{M} \), the curves \( u^j = \text{sbt.} \ 2 \leq j \leq n \), are the integral curves of the vectorfield \( x_T \). Thus \( g_{ij} = b_{ij} = 0 \) and \( \bar{k}_i = \frac{b_{ii}}{g_{ii}} \). Moreover, \( r = r(u^1) \) and \( f = f(u^1) \) because \( x_T \) is parallel \( x_i \). We can write the position vector \( x \) of \( M \) with respect to the basis \( \{x_1, \ldots, x_n, N\} \) of \( E^{n+1} \),
\[ x = \sum_{i=1}^{n} c_i x_i + c_{n+1} N. \]

We compute the coefficients, \( c_i = \frac{\langle x, x_i \rangle}{g_{ii}} = \frac{\rho_i}{g_{ii}} \) and \( c_{n+1} = -f \). Since \( r_i = 0 \), \( i \neq 1 \), we obtain
\[ x = \frac{\rho_1}{g_{11}} x_1 - f N. \quad (3.2) \]

From (1.3) and (3.2)
\[ x_T = \frac{\rho_1}{g_{11}} x_1, \quad \rho_1 = \frac{\sqrt{g_{11}}}{|x_T|}. \]

Since \( |x_T|^2 = r^2 - f^2 \), the \( g_{11} \) depends on \( u^1 \) only. We differentiate (3.2) with respect to \( u^1 \).
\[ x_i = \frac{\pi_i}{g_{ii}} \cdot x_{ii} - f \cdot N_i, \quad 2 \leq i \leq n. \]

Using Rodrigues formula \( \mathbf{N} = -kx \), we get
\[
\left( \frac{1 - fk}{|x_i|} \right) x_i = \frac{1}{\sqrt{g_{11}}} x_{ii}.
\]

If we product both sides of the last equation with \( x_j \) then
\[
\left( \frac{1 - fk}{|x_i|} \right) = \frac{1}{2\sqrt{g_{11}}} \frac{\partial}{\partial u^i} \left( \log g_{ii} \right), \quad 2 \leq i \leq n.
\]

Since \( \mathbf{N} = -\frac{x_i}{|x_i|} \), we can take as \( \mathbf{N} = -\frac{x_i}{\sqrt{g_{11}}} \). Set \( h = \frac{f}{r} \), then \( x = hx \) and
\[
\bar{x}_i = h_i x + hx_i,
\]

where \( h_i = -\frac{h \sqrt{g_{11}}}{|x_i|} \) and \( h_i = 0, \quad 2 \leq i \leq n \). Thus, \( \bar{g}_{ii} = \frac{h^2 f^2}{|x_i|} g_{ii}, \quad \bar{g}_{ij} = 0 \), \( i \neq j \).

\[ \bar{g}_{ii} = h^2 g_{ii}, \quad 2 \leq i \leq n. \] Similarly \( \bar{b}_{ii} = \frac{h f}{|x_i|} b_{ii}, \quad i \neq j, \quad \bar{b}_{ii} = \frac{h}{2 \sqrt{g_{11}}} \frac{\partial g_{ii}}{\partial u^1}, \]

\[ 2 \leq i \leq n. \] Therefore, the parameter curves of \( \bar{M} \) are the lines of curvature, so that the natural mapping preserves principal directions.

For the principal curvatures of \( M \), we obtain
\[ \bar{k}_i = \frac{\bar{b}_{ii}}{\bar{g}_{ii}} = \frac{f^2 f}{h^2} k_i, \]

and
\[ \bar{k}_i = \frac{\bar{b}_{ii}}{\bar{g}_{ii}} = \frac{1}{2h \sqrt{g_{11}}} \frac{\partial}{\partial u^1} \left( \log g_{ii} \right) = \frac{1 - fk}{f}, \quad 2 \leq i \leq n. \]

This completes the proof.

**Corollary.** The Gauss curvature of \( \bar{M} \) is
\[
\bar{K} = \frac{k_i}{h^2 f^2 f^2} \left[ 1 - f \sum_{i=2}^{n} k_i + f^2 \sum_{i=2}^{n} k_i k_j - f^3 \sum_{i=2}^{n} k_i k_j k_l + \ldots \right]
\]
\[ \ldots - f^{n-2} \sum_{i=2}^{n} k_i \ldots k_n \nonumber \] + \[ \frac{f^{n-2}}{h f} \]

in which \( K \) is the Gauss curvature of \( M \) are \( \hat{k}_i \) is meant dropping i-th curvature function \( k_i \) of \( M \).
Corollary 3.5. The mean curvature of $\overline{M}$ is
\[
\overline{H} = \frac{(n - 1) f + r^2 k_1}{nf} \cdot \int_{\mathcal{F}} H,
\]
where $H$ is the mean curvature of $M$.

REFERENCES

