ON SEPARATION AXIOM C-D₁

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1. INTRODUCTION

In 1997, Caldas [1] has introduced a new separation axiom semi-D₁ which is situated between semi-T₀ and semi-T₁ due to Maheshwari and Prasad [5]. In 1996, Hatir, Noiri and Yüksel [2] defined C-sets and C-continuity in topological spaces to obtain a decomposition of continuity. Quite recently, Jafari [3] has used the C-sets to define and investigate C-T₂ spaces, C-compact spaces and C-connected spaces. In this paper, we define cD-sets as the difference set of C-sets and use these sets to define C-D₁-spaces, cD-compact spaces and cD-connected spaces. We also investigate the relationship between these spaces and C-continuity (or C-irresoluteness).

2. PRELIMINARIES

Throughout this paper X and Y denote topological spaces on which no separation axiom is assumed. Let A be a subset of a space X. The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively.

We shall recall some definitions used in the sequel.

Definition 2.1. A subset A of a space X is said to be

(a) semi-open [4] if \( A \subseteq \text{Cl} (\text{Int}(A)) \),
(b) \( \alpha^* \)-set [2] if \( \text{Int} (\text{Cl} (\text{Int}(A))) = \text{Int}(A) \),
(c) C-set [2] if \( A = \text{O} \cap F \), where O is open and F is an \( \alpha^* \)-set.
Remark 2.1. Semi-open sets and C-sets are independent. A set \( \{a, b\} \) in [2, Example 3.1] is a C-set but it is not semi-open. A set \( \{a, b\} \) in Example 3.1 (below) is semi-open but it is not a C-set.

Definition 2.2. A function \( f: X \to Y \) is said to be \( C \)-continuous [2] (resp. \( semi \)-continuous [4]) for each open set \( V \) of \( Y \), \( f^{-1}(V) \) is a C-set (resp. \( semi \)-open in \( X \).

3. \( C-D_1 \) SPACES

Definition 3.1. A subset \( S \) of a space \( X \) is called a \( c \)-difference (briefly \( cD \)-set) (resp. \( D \)-set [6], \( sD \)-set [1]) if there exist two C-sets (resp. open sets, semi-open sets) \( O_1, O_2 \) in \( X \) such that \( O_1 \neq X \) and \( S = O_1 \setminus O_2 \).

Remark 3.1. Every proper C-set is a \( cD \)-set, but the converse is false as the following example shows.

Example 3.1. Let \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, X, \{a\}, \{a, d\}, \{A, b, d\}, \{a, c, d\}\} \). Then \( \{a, b\} \) is a \( cD \)-set but it is not a C-set.

Definition 3.2. A topological space \( X \) is \( C-D_0 \) (resp. \( C-D_1 \)) if for \( x, y \in X \) such that \( x \neq y \) there exists a \( cD \)-set of \( X \) containing \( x \) but not \( y \) or (resp. and) a \( cD \)-set containing \( y \) but not \( x \).

A topological space \( X \) is \( C-T_0 \) (resp. \( C-T_1 \)) if for \( x, y \in X \) such that \( x \neq y \) there exists a C-set of \( X \) containing \( x \) but not \( y \) or (resp. and) a C-set containing \( y \) but not \( x \).

Definition 3.3. A topological space \( X \) is \( C-D_2 \) (resp. \( C-T_2 \) [3]) if for \( x, y \in X \) such that \( x \neq y \) there exist disjoint \( cD \)-sets (resp. C-sets) \( S_1 \) and \( S_2 \) such that \( x \in S_1 \) and \( y \in S_1 \).

Remark 3.2. The following implications hold:

a) If \( X \) is \( T_i \), then \( X \) is \( C-T_i \), for \( i = 0, 1, 2 \).

b) If \( X \) is \( C-T_i \), then \( X \) is \( C-D_i \), for \( i = 0, 1, 2 \).

b) If \( X \) is \( C-D_i \), then \( X \) is \( C-D_i \), for \( i = 1, 2 \).

d) If \( X \) is \( C-T_{i-1} \), then \( X \) is \( C-T_{i-1} \), for \( i = 1, 2 \).
Theorem 3.1. A topological space $X$ is C-D$_0$ if and only if it is C-T$_0$.

Proof. The sufficiency is Remark 3.2 (b).

Necessity: Let $X$ be C-D$_0$. Then for each pair of distinct points $x, y \in X$, at least one of $x, y$, say $x$, belongs to a cD-set $S$ but $y \notin S$. Let $S \in O_1 \setminus O_2$, where $O_1 \neq X$ and $O_1$ and $O_2$ are C-sets. Then $X \in O_1$ and for $y \notin S$ we have two cases:

(1) $y \notin O_1$; (2) $y \in O_1$ and $y \in O_2$

In case (1): $O_1$ contains $x$ but doesn't contain $y$.
In case (2): $O_2$ contains $y$ but doesn't contain $x$. Thus $X$ is C-T$_0$.

Theorem 3.2. If a topological space $X$ is C-D$_1$, then it is C-T$_0$.

Proof. This follows from Remark 3.2 and Theorem 3.1.

Theorem 3.3. If $f : X \to Y$ is a semi-continuous (resp. C-continuous surjection and $S$ is a D-set in $Y$, then $f^{-1}(S)$ is a sD-set (resp. cD-Set) in $X$.

Proof. We prove only the first case being the second similar. Let $S$ be a D-set of $Y$. Then there are open sets $O_1$ and $O_2$ in $Y$ such that $S = O_1 \setminus O_2$ and $O_1 \neq Y$. By the semi-continuity of $f$, $f^{-1}(O_1)$ and $f^{-1}(O_2)$ are semi-open in $X$. Since $O_1 \neq Y$ and $f$ is surjective, we have $f^{-1}(O_1) = X$. Hence $f^{-1}(S) = f^{-1}(O_1) \cup f^{-1}(O_2)$ is a sD-set.

A space $X$ is said to be semi-D$_1$ [1] if for any pair of distinct points $x$ and $y$ of $X$, there exist sD-sets $U$ and $V$ of $X$ such that $x \in U$, $y \notin U$, $x \notin V$ and $y \in V$.

Theorem 3.4. If $y$ is a D$_1$-space and $f : X \to Y$ a is semi-continuous (resp. C-continuous) bijection, then $X$ is a semi-D$_1$ (resp. C-D$_1$) space.

Proof. We prove only the first case being the second is analogous. Suppose that $Y$ is a D$_1$-space. Let $x$ and $y$ be any pair of distinct points in $X$. Since $f$ is injective and $Y$ is D$_1$-space, there exist D-sets $S_x$ and $S_y$.
of $Y$ containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin S_x$, $f(x) \notin S_y$. By Theorem 3.3, $f^{-1}(S_x)$ and $f^{-1}(S_y)$ are sD-sets in $X$ containing $x$ and $y$ respectively, such that $y \notin f^{-1}(S_x)$ and $x \notin f^{-1}(S_y).$ This implies that $X$ is a semi-$D_1$ space.

**Definition 3.4.** A function $f : X \rightarrow Y$ is called $C$-irresolute if for every $C$-set $A$ in $Y$, its inverse image $f^{-1}(A)$ is $C$-set in $X$.

**Theorem 3.5.** If $f : X \rightarrow Y$ is a $C$-irresolute surjection and $S$ is a cD-set of $Y$, then $f^{-1}(S)$ is a cD-set of $X$.

**Proof.** Suppose that $S$ is a cD-set of $Y$. Then there are $C$-sets $O_1$ and $O_2$ in $Y$ such that $S = O_1 \setminus O_2$ and $O_1 \neq Y$. By the $C$-irresoluteness of $f$, $f^{-1}(O_1)$ and $f^{-1}(O_2)$ are $C$-sets in $X$. Since $O_1 \neq Y$, we have $f^{-1}(O_1) \neq X$. Hence $f^{-1}(S) = f^{-1}(O_1) \setminus f^{-1}(O_2)$ is a cD-set.

**Theorem 3.6.** A space $X$ is $C$-$D_1$ if and only if for each pair of distinct points $x$ and $y$ of $X$, there exist a $C$-irresolute surjection $f$ of $X$ onto a $C$-$D_1$ space $Y$ such that $f(x) \neq f(y)$.

**Proof.** Necessity: Take the identity function on $X$.

Sufficiency: Let $x$ and $y$ be any pair of distinct points in $X$. By hypothesis, there exists a $C$-irresolute surjection $f$ of $X$ onto a $C$-$D_1$ space $Y$ such that $f(x) \neq f(y)$. Therefore, there exist cD-sets $S_x$ and $S_y$ in $Y$ such that $f(x) \in S_x$, $f(y) \notin S_x$, $f(y) \in S_y$, $f(x) \notin S_y$. Since $f$ is $C$-irresolute and surjective, by Theorem 3.5, $f^{-1}(S_x)$ and $f^{-1}(S_y)$ are cD-sets in $X$ such that $x \notin f^{-1}(S_x)$, $y \notin f^{-1}(S_y)$; $y \in f^{-1}(S_y)$, $x \notin f^{-1}(S_y)$. Therefore, $X$ is a $C$-$D_1$ space.

We can give the following notions:

**Definition 3.5.** A filterbase $B$ is called cD-convergent (resp. D-convergent) to a point $x \in X$ if for any cD-set (resp. D-set) $A$ containing $x$, there exists $B \in B$ such that $B \subseteq A$.

**Theorem 3.7.** If function $f : X \rightarrow Y$ is $C$-continuous and surjective, then for each point $x \in X$ and each filterbase $B$ on $X$ cD-convergent to $x$, the filterbase $f(B)$ is D-convergent to $f(x)$. 
Proof. Let \( x \in X \) and \( B \) be any filterbase cD-convergent to \( x \). Since \( f \) is a C-continuous surjection, by Theorem 3.3, for each D-set \( V \subset Y \) containing \( f(x) \), \( f^{-1}(V) \subset X \) is a cD-set containing \( x \). Since \( B \) is cD-convergent \circ x \), then there exists \( B \in B \) such that \( b \subset f^{-1}(V) \); hence \( f(B) \subset V \). It follows that \( f(B) \) is d-convergent to \( f(x) \).

Corollary 3.1. If a function \( f : X \to Y \) is C-irresolute and surjective, then for each point \( x \in X \) and each filterbase \( B \) on \( X \) cD-convergent to \( x \), filterbase \( f(B) \) is cD-convergent to \( f(x) \).

We can give the following notions:

Definition 3.6. A space \( X \) is called cD-compact (resp. D-compact) if every cover of \( X \) by cD-sets (resp. D-sets) has a finite subcover.

Theorem 3.8. Let a function \( f : X \to Y \) be C-continuous and surjective. If \( X \) is cD-compact, then \( Y \) is D-compact.

Proof. Let \( \gamma \) be an cover of \( Y \) by D-sets. Since \( f \) is C-continuous and surjective, by Theorem 3.3, \( f^{-1}(\gamma) = \{ f^{-1}(V_\gamma), V \in \gamma \} \) is a cover of \( X \) by cD-sets. Since \( X \) is cD-compact, there exists a finite subcover \( \{ f^{-1}(V_\gamma), ..., f^{-1}(V_n) \} \) of \( f^{-1}(\gamma) \). Therefore, \( \{ V_1, ..., V_n \} \) is a finite subcover of \( g \). Hence \( Y \) is D-compact.

Corollary 3.2. Let \( f : X \to Y \) be a C-irresolute surjection. If \( X \) is cD-compact, then \( Y \) is cD-compact.

We can also give the following notion.

Definition 3.7. A space \( X \) is called cD-connected (resp. D-connected) if \( X \) can not be expressed as the union of two nonempty disjoint cD-sets (resp. D-sets).

Theorem 3.9. If \( f : X \to Y \) is a C-continuous surjection and \( X \) is cD-connected, then \( Y \) is D-connected.

Proof. Straightforward.

Corollary 3.3. If \( f : X \to Y \) is a C-irresolute surjection and \( X \) is cD-connected, then \( Y \) is cD-connected.
REFERENCES


