ON THE INEQUIVALENCE AND STANDARD BASIS OF THE SPECHT MODULES OF THE HYPEROCTAHEDRAL GROUPS

H. CAN

Department of Mathematics, Faculty of Arts & Science, Erciyes University, Kayseri, TURKEY

(Received Feb. 23, 1998; Accepted June 22, 1998)

ABSTRACT

The representations of the hyperoctahedral groups $O_n$ has been studied by many authors, see for example Al-Aamly, Morris and Peel and Morris. The latter author has interpreted the work of the first three authors in the combinatorial language used in the representation theory of the symmetric groups, but a work on the inequivalence and standard basis of the Specht modules of $O_n$ has not yet appeared in the literature. Therefore, in this paper we show that Specht modules of the hyperoctahedral groups are mutually non-isomorphic and determine the standard basis of the Specht modules.

1. INTRODUCTION

An account of the irreducible representations of the hyperoctahedral groups $O_n$ can be found in a variety of places, see for example Mayer [3], Al-Aamly, Morris and Peel [1] and Morris [4]. The first author has constructed simple left ideals in the classical case for these groups. The last author has translated the work of [1] to the language of the combinatorial concepts used in the representation theory of the symmetric groups. We first establish the basic notation and state some results which are required later. We refer the reader to [1] and [4] for much of the undefined terminology.

The hyperoctahedral group $O_n$ (or $C_n$ in the notation of Weyl groups) is the group of all permutations $\sigma$ of $\{\pm 1, \pm 2, \ldots, \pm n\}$ such that $\sigma(-i) = -\sigma(i)$ for $i = 1, 2, \ldots, n$. A positive transposition has the form $(a, b)(-a, -b)$. These generate a subgroup of $O_n$ isomorphic to the symmetric group $S_n$. A negative transposition is of the form $(a, -a)$. These generate a normal subgroup isomorphic to $C_2 \times C_2 \times \cdots \times C_2$ ($n$ factors).
A pair of partitions of $n$ $(\lambda, \mu)$ consists of partitions $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ and $\mu = (\mu_1, \mu_2, ..., \mu_s)$ such that $\lambda$ is a partition of $|\lambda|$ and $\mu$ is a partition of $|\mu|$, and $|\lambda| + |\mu| = n$. A double Young diagram \cite{1} is defined in the obvious way. A partial order on the set of pairs of partitions of $n$ may be defined as follows. Let $(\lambda, \mu)$ and $(\lambda', \mu')$ be pairs of partitions of $n$. $(\lambda, \mu)$ dominates $(\lambda', \mu')$, written $(\lambda, \mu) \triangleright (\lambda', \mu')$ if $|\lambda| > |\lambda'|$ or if $|\lambda| = |\lambda'|$, $|\mu| = |\mu'|$ with $\lambda \triangleright \lambda'$ and $\mu \triangleright \mu'$.

Let $(\lambda, \mu)$ be a pair of partitions of $n$. A $(\lambda, \mu)$-tableau $t$ is an array of integers obtained by replacing each node in $[\lambda, \mu]$ by one of the integers $\pm 1$, $\pm 2$, ..., $\pm n$ with $i$ and $-i$ ($i = 1$, ..., $n$) not appearing simultaneously. A $(\lambda, \mu)$-tableau $t$ will be sometimes written $(t^\lambda, t^\mu)$. Let

$$t(i, j, k) = \begin{cases} t^\lambda(i, j) & \text{if } k = 1 \\ t^\mu(i, j) & \text{if } k = 2, \end{cases}$$

where $t^\lambda(i, j)$ and $t^\mu(i, j)$ stand for the entry of $t^\lambda$ and $t^\mu$ in in position $(i, j)$, as in a matrix, respectively. Let $R_t$ (resp. $C_t$) be the group of row (resp. column) permutations of $t$.

Two, $(\lambda, \mu)$-tableaux $t_1$ and $t_2$ are row equivalent, written $t_1 \sim t_2$, if there exists $\sigma \in R_t$ such that $t_2 = \sigma t_1$. The equivalence class which contains the $(\lambda, \mu)$-tableau $t$ is $\{t\}$ and is called a $(\lambda, \mu)$-tabloid. The Group $O_n$ acts on the set of $(\lambda, \mu)$-tabloids by $\sigma\{t\} = \{\sigma t\}$ for all $\sigma \in O_n$.

Let $K$ be an arbitrary field. If $(\lambda, \mu)$ is a pair of partitions of $n$, let $M^{\lambda, \mu}$ be the vector space over $K$ spanned by $(\lambda, \mu)$-tabloids. By extending the above action linearly to $KO_n$, we have that $M^{\lambda, \mu}$ is a $KO_n$-module. As in the case of the symmetric group (see [2]), a bilinear form $\langle ..,.. \rangle$ on $M^{\lambda, \mu}$ is defined in the obvious way.

Let $t$ be a $(\lambda, \mu)$-tableau. Define $\kappa \in KO_n$ by $\kappa_t = \sum_{\sigma \in S_t} (\text{sgn } \sigma)\sigma$, where $\text{sgn } (\sigma) = (-1)^{\ell(\sigma)}$ is the sign function and $\ell(\sigma)$ is the length of $\sigma$. The $(\lambda, \mu)$-polytabloid $e_t$ associated with the tableau $t$ is given by $e_t = \kappa_t\{t\}$. The Specht module $S^{\lambda, \mu}$ for the pair of partitions $(\lambda, \mu)$ is the submodule of $M^{\lambda, \mu}$ spanned by $(\lambda, \mu)$-polytabloids.
1.1. Let $t$ be a $(\lambda, \mu)$-tableau and $t'$ be a $(\lambda', \mu')$-tableau such that for all $a$, if $a$ occurs in $t_\lambda$ then $\pm a$ occurs in $t'_\lambda$. Suppose that $a, b$ belong to the same row of $t'_\lambda$, implies that $c, d$ belong to different columns of $t_\gamma$, where $c = \pm a, d = \pm b$ and $\gamma = \lambda$ or $\mu$. Then $(\lambda, \mu) \leq (\lambda', \mu')$.

1.2. Let $t$ be a $(\lambda, \mu)$-tableau and $\sigma \in O_n$. Then we have the following facts:

(i) $e_{\sigma t} = \sigma e_t$, and if $\sigma \in C_t$ then $s e_t = (\text{sgn } \sigma)e_t$.

(ii) If $u \in M^{\lambda, \mu}$ then $\kappa_t u$ is a multiple of $e_t$.

(iii) Let $t'$ be another $(\lambda', \mu')$-tableau. If there exist $a, b$ in the same row of $t'_\gamma$, such that $c, d$ are in the same column of $t_\gamma$, where $c = \pm a, d = \pm b$ and $\gamma = \lambda$ or $\mu$, then $\kappa_t \{t'\} = 0$.

1.3. If $U$ is a submodule of $M^{\lambda, \mu}$ than either $U \supseteq S^{\lambda, \mu}$ or $U \subseteq (S^{\lambda, \mu})^\perp$, where $(S^{\lambda, \mu})^\perp$ is the complement of $S^{\lambda, \mu}$ in $M^{\lambda, \mu}$. Furthermore, the module $S^{\lambda, \mu}/S^{\lambda, \mu} \cap (S^{\lambda, \mu})^\perp$ is zero or irreducible.

2. INEQUIVALENCE

In this section, we prove that Specht modules of $O_n$ are mutually non-isomorphic.

**Lemma 2.1.** Let $t$ be a $(\lambda, \mu)$-tableau. Then

(i) $\kappa_t \kappa_t = |C_t| \kappa_t$, 
(ii) $\kappa_t e_t = |C_t| e_t$.

**Proof.** We know that $\kappa_t = \sum_{\sigma \in C_t} (\text{sgn } \sigma)\sigma$. Hence,

$$\kappa_t \kappa_t = \left( \sum_{\sigma \in C_t} (\text{sgn } \sigma)\sigma \right) \left( \sum_{\pi \in C_t} (\text{sgn } \pi)\pi \right)$$

$$= \sum_{\sigma, \pi \in C_t} (\text{sgn } \sigma)(\text{sgn } \pi)\sigma\pi$$

$$= \sum_{\sigma, \pi \in C_t} (\text{sgn } \sigma)(\text{sgn } \sigma^{-1})\gamma \gamma$$ (where $\sigma\pi = \gamma$)

$$= \sum_{\sigma \in C_t} \sum_{\gamma \in C_t} (\text{sgn } \gamma)\gamma = |C_t|\kappa_t, \text{ proving (i)}$$
(ii) follows from (i)

**Lemma 2.2.** Let \( t \) be a \((\lambda, \mu)\)-tableau and \( t' \) be a \((\lambda', \mu')\)-tableau. Suppose that \( |\lambda| > |\lambda'| \). Then we have the following facts:

(i) \( \kappa_1(t') = 0 \)

(ii) \( \kappa_1 \epsilon_r = 0. \)

**Proof.** Since \( |\lambda| > |\lambda'| \) then there exists at least one entry \( a \) in \( t \) such that \( \pm a \) occurs in \( t_{\lambda.} \). Then \( (a, -a) \in C_t \cap R_{t'} \) Thus, \( (e-(a, -a)) \{t'\} = 0 \). Since \( (a, -a) \in C_t \) then \( (a, -a) \) generates a subgroup of order 2 of \( C_t \) Select signed coset representatives \( \sigma_1, \ldots, \sigma_k \) for this subgroup of \( C_t \); then

\[
\kappa_1(t') = \left( \sum_{i=1}^{k} \sigma_i \right) (e - (a, -a)) \{t'\} = 0 \quad \text{as required.}
\]

(ii) follows immediately from (i).

**Lemma 2.3.** Let \( t \) be \((\lambda, \mu)\)-tableau and \( t' \) a \((\lambda', \mu')\)-tableau such that for all \( a \), if \( a \) occurs in \( t \) then \( \pm a \) occurs in \( t_{\lambda'} \). Suppose that \( \kappa_1(t') \neq 0 \). Then \((\lambda, \mu) \supset (\lambda', \mu') \).

**Proof.** Suppose that \( a \) and \( b \) are two elements in the same row of \( t_{\gamma} \), where \( \gamma = \lambda' \) or \( \mu' \). Then \( c \) and \( d \) cannot be in the same column of \( t_{\gamma} \), where \( c = \pm a \), \( d = \pm b \) and \( \gamma = \lambda \) or \( \mu \), for if so, then by (1.2) (iii) \( \kappa_1(t') = 0 \), contradicting our hypothesis. Thus, (1.1) yields \((\lambda, \mu) \supset (\lambda', \mu') \).

**Proposition 2.4.** Suppose that the field of scalars is \( Q \) and \( \theta \in \text{Hom}(S^{\lambda, \mu}, S^{\lambda', \mu'}) \) where \( |\lambda| > |\lambda'| \). Then \( \theta = 0 \).

**Proof.** Let \( \theta \in \text{Hom}(S^{\lambda, \mu}, S^{\lambda', \mu'}) \) where \( |\lambda| > |\lambda'| \). Let \( t \) be a \((\lambda, \mu)\)-tableau and \( t' \) be a \((\lambda', \mu')\)-tableau. Then

\[
\theta(e_t) = \sum \alpha_{t'} e_{t'},
\]

where the summation is taken over all \((\lambda', \mu')\)-tableaux. By Lemma 2.1 and Lemma 2.2 we have

\[
|\mathcal{C}_t| \theta(e_t) = \theta(\kappa_1 e_t) = \kappa_1 \theta(e_t) = \sum \alpha_{t'} \kappa_1 e_{t'} = 0.
\]

Thus \( \theta(e_t) = 0 \) and \( \theta = 0 \) as required, which implies that \( S^{\lambda, \mu} \) and \( S^{\lambda', \mu'} \) are not isomorphic when \( |\lambda| > |\lambda'| \).
Proposition 2.5. Suppose the field of scalars is \( \mathbb{Q} \) and \( \theta \in \text{Hom} (S^{\lambda, \mu}, M^{\lambda', \mu'}) \) is non-zero, where \( |\lambda| > |\lambda'| \). Thus, \((\lambda, \mu) \triangleright (\lambda', \mu')\) and if \((\lambda, \mu) = (\lambda', \mu')\) then \(\theta\) is multiplication by a scalar.

Proof. Since \(\theta \neq 0\), there is some basis vector \(e_t\) such that \(\theta(e_t) \neq 0\). Because \(\langle , , \rangle\) is an inner product with rational scalars, \(M^{\lambda, \mu} = S^{\lambda, \mu} \oplus (S^{\lambda, \mu})^\perp\). Thus, we can extend \(\theta\) to an element of \(\text{Hom}(M^{\lambda, \mu}, M^{\lambda', \mu'})\) by setting \(\theta((S^{\lambda, \mu})^\perp) = 0\). So

\[
0 = \theta(e_t) = \theta(\kappa_i(1, t)) = \kappa_i \theta(t) = \kappa_i \left( \sum \alpha_i(t_{i_j}) \right)
\]

where the \(t_{i_j}\) are \((\lambda', \mu')\)-tableaux.

By Lemma 2.3 we have \((\lambda, \mu) \triangleright (\lambda', \mu')\). In the case \((\lambda, \mu) = (\lambda', \mu')\), (1.2) (ii) yields \(\theta(e_t) = \alpha e_t\) for some scalar \(\alpha\). So for any \(\sigma \in \mathfrak{S}_n\),

\[
\theta(e_{\alpha_t}) = \theta(\sigma e_t) = \sigma \theta(e_t) = \sigma(\alpha e_t) = \alpha e_{\sigma t}.
\]

Thus \(\theta\) is multiplication by a scalar.

We can finally verify all our claims about the Specht modules.

Theorem 2.6. Let \((\lambda, \mu)\) be a pair of partitions of \(n\) and \(K = \mathbb{Q}\). Then the \(S^{\lambda, \mu}\) give a complete set of irreducible \(K\mathfrak{S}_n\)-modules.

Proof. The \(S^{\lambda, \mu}\) are irreducible by (1.3) and the fact that \(S^{\lambda, \mu} \cap (S^{\lambda, \mu})^\perp = \{0\}\) for the field \(\mathbb{Q}\). Since we have the right number of modules for a full set, it suffices to show that they are pairwise inequivalent.

Now, let \((\lambda, \mu)\) and \((\lambda', \mu')\) be pairs of partitions of \(n\). If \(|\lambda| > |\lambda'|\) then by Proposition 2.4, \(S^{\lambda, \mu}\) and \(S^{\lambda', \mu'}\) are not isomorphic. If \(|\lambda| = |\lambda'|\) and \(S^{\lambda, \mu} \cong S^{\lambda', \mu'}\) then there exists a non-zero homomorphism \(\theta \in \text{Hom} (S^{\lambda, \mu}, M^{\lambda', \mu'})\) since \(S^{\lambda', \mu'} \subseteq M^{\lambda', \mu'}\). Thus by Proposition 2.5, \((\lambda, \mu) \triangleright (\lambda', \mu')\). Similarly, \((\lambda', \mu') \triangleright (\lambda, \mu)\), so that \((\lambda, \mu) = (\lambda', \mu')\).

3. THE STANDARD BASIS OF THE SPECHT MODULES

In this section, we determine the standard basis of the Specht modules \(S^{\lambda, \mu}\).
**Definition 3.1.** A $(\lambda, \mu)$-tableau $t$ is said to be standard if the entries of $t$ are all positive integers, and $t_\lambda$ and $t_\mu$ are both standard tableaux. A tabloid $\{t\}$ is a standard tabloid if there is a standard tableau in the equivalence class $\{t\}$. A polytabloid $e_t$ is a standard polytabloid if $t$ is a standard tableau.

**Theorem 3.2.** The set $\{e_t : t$ is a standard $(\lambda, \mu)$-tableau$\}$ is a $K$-basis for $S^{\lambda, \mu}$. The dimension of $S^{\lambda, \mu}$ is the number of standard $(\lambda, \mu)$-tableaux.

We will spend this section proving this theorem. In the first place we will establish that the $e_t$ are linearly independent. For this case, we shall need a partial order on tabloids associated with the same pair of partitions of $n$.

Let $t = (t_\lambda, t_\mu)$ be a $(\lambda, \mu)$-tableau. Let $|t_\mu|$ denote the modulus of $t_\mu$, that is, if $t_\mu = (t(i,j))$, where $t(i,j)$ stands for the entry of $t_\mu$ in position $(i,j)$ then $|t_\mu| = \{t(i,j)\}$.

For example, if

$$
t_\mu = \begin{pmatrix}
-1 & 2 & 5 \\
4 & -6 & \\
-3 &
\end{pmatrix}
$$

then $|t_\mu| = \{1, 2, 5\}$.

Given any tableau $t = (t_\lambda, t_\mu)$, let $m_{ir}(t_\lambda)$ denote the number of entries less than or equal to $i$ for $i = \pm 1, \ldots, \pm n$ in the first $r$ rows of $t_\lambda$, and let $n_{ir}(t_\lambda)$ denote the number of entries less than or equal to $i$ for $i = 1, \ldots, n$ in the first $r$ rows of $|t_\mu|$.

**Definition 3.3.** Let $(\lambda, \mu)$ be a pair of partitions of $n$. Let $\{t\}$ and $\{s\}$ be $(\lambda, \mu)$-tabloids. Then we write

(a) $\{t_\lambda\} \preceq \{s_\lambda\}$ if one of the following conditions is satisfied:

1. some entries of $t_\lambda$ are negative and the entries in $s_\lambda$ are all positive (in this case $\{t_\lambda\} \prec \{s_\lambda\}$),

2. for all $i$ $(i = \pm 1, 2, \ldots, \pm n)$ and $r$, $m_{ir}(t_\lambda) \leq m_{ir}(s_\lambda)$, and

(b) $\{t_\mu\} \preceq \{s_\mu\}$ if for all $i$ $(1 \leq i \leq n)$ and $r$, $n_{ir}(t_\mu) \leq n_{ir}(s_\mu)$. 

Thus, \( \{s\} \) dominates \( \{t\} \), written \( \{t\} \not\subseteq \{s\} \), if \( \{t,\} \not\subseteq \{s,\} \) and \( \{t_\mu\} \not\subseteq \{s_\mu\} \).

Let \( t \) be a \((\lambda, \mu)\)-tableau. Suppose that \( a \) is in the \( p\)-th row and \( b \) is the \( q\)-th row of \( t_\gamma \), where \( a = \pm k \), \( b = \pm \ell \) \((1 \leq k, \ell \leq n)\) and \( \gamma = \lambda \) or \( \mu \). If \( \gamma = \lambda \) and \( a < b \) then the definition of \( m_{ir}(t_\lambda) \) gives

\[
m_{ir}(a, b)(-a, -b)t_\lambda - m_{ir}(t_\lambda) = \begin{cases} 
1 & \text{if } q \leq r < p \text{ and } a \leq i < b \\
-1 & \text{if } p \leq r < q \text{ and } a \leq i < b \\
0 & \text{otherwise}
\end{cases}
\]

If \( \gamma = \mu \) and \( k < \ell \) then the definition of \( n_{ir}(t_\mu) \) gives

\[
n_{ir}(a, b)(-a, -b)t_\mu - n_{ir}(t_\mu) = \begin{cases} 
1 & \text{if } q \leq r < p \text{ and } k \leq i < \ell \\
-1 & \text{if } p \leq r < q \text{ and } k \leq i < \ell \\
0 & \text{otherwise}
\end{cases}
\]

Therefore we have the following lemma.

**Lemma 3.4.** (Dominance Lemma for Tabloids) Let \( \{t\} \) be a \((\lambda, \mu)\)-tabloid. Suppose there exist \( a, b \) in \( t \) such that \( a \) appears in a lower row than \( b \), where \( a = \pm k \), \( b = \pm \ell \) \((1 \leq k, \ell \leq n)\). Then \( \{t\} \not\subseteq (a, b)(-a, -b)\{t\} \) if one of the following conditions is satisfied:

(i) \( \gamma = \lambda \) and \( a < b \),

(ii) \( \gamma = \mu \) and \( k < l \).

The previous lemma puts a restriction on which tabloids can appear in a standard polytabloid, that is

**Corollary 3.5.** If \( t \) is standard and \( \{s\} \) appears in \( e_t \), then \( \{s\} \not\subseteq \{t\} \).

**Proof.** Let \( s = \pi t \), where \( \pi \in C_t \), so that \( \{s\} \) appears in \( e_t \). We induct on the number of column inversions in \( s \). We know that a column permutation of \( t \) permutes the elements in each column of \( t \) and may change the sign of elements in \( t_\lambda \). Thus we consider all possible cases in terms of \( \pi \in C_t \):

Let \( \pi = \sigma \tau \) be such that \( \tau \in S_n \) and \( s = \prod_i w_i \in C_{2}^{n} \), where each \( i \) appears in \( t_\lambda \).
(i) Firstly, if \( \tau = e \) then \( \pi = \sigma \). Thus, since \( s = \sigma t \), \( \sigma \) changes the sign of some elements in \( t_\lambda \) and the entries in \( t_\mu \) do not change. Then \( \{s_\lambda\} \not< \{t_\lambda\} \) by part (a) (1) of Definition 3.3 and \( \{s_\mu\} = \{t_\mu\} \). Therefore, \( \{s\} \not< \{t\} \).

(ii) Secondly, if \( \sigma = e \) then \( \pi = \tau \). Thus, since \( s = \tau t \), then in the same column of \( s_\gamma \) there are positive integers \( a < b \) such that \( a \) is in a lower row than \( b \), where \( \gamma = \lambda \) or \( \mu \). Thus, by Lemma 3.4 \( \{s\} \not< \{a, b\} \) \(-a, -b\) \{s\}. Since \( \{a, b\} \) \(-a, -b\) \{s\} is involved in \( e_t \) and \( \{a, b\} \) \(-a, -b\) \{s\} has fewer inversions than \{s\}, induction shows that \( \{a, b\} \) \(-a, -b\) \{s\} \not< \{t\}. Therefore, \( \{s\} \not< \{t\} \).

(iii) Finally, if \( \tau, \sigma \neq e \) then \( \pi = \sigma \tau \) and \( s = \pi t \). By (i), some entries of \( s_\lambda \) are negative. Then we have at once \( \{s_\lambda\} \not< \{t_\lambda\} \) by part (a) (1) of Definition 3.3. On the other hand, \( \{s_\mu\} \not< \{t_\mu\} \) by (ii). Therefore \( \{s\} \not< \{t\} \). Hence, we have the required result.

The previous corollary says that \( \{t\} \) is the maximum tabloid in \( e_t \).

**Lemma 3.6.** Let \( v_1, \ldots, v_k \) be elements of \( M_{\lambda \mu}^\lambda \). Suppose, for each \( v_i \), we can choose a tabloid \( \{t_i\} \) appearing in \( v_i \) such that

1. \( \{t_i\} \) is maximum in \( v_i \), and
2. the \( \{t_i\} \) are all distinct.

Then \( v_1, \ldots, v_k \) are linearly independent over \( K \).

**Proof.** Choose the labels such that \( \{t_1\} \) is maximal among the \( \{t_i\} \). Thus conditions 1 and 2 ensure that \( \{t_1\} \) appears only in \( v_1 \). If \( \{t_i\} \) occurs in \( v_i \), \( i > 1 \), then \( \{t_i\} \not< \{t_1\} \), contradicting the choice of \( \{t_1\} \). It follows that in any linear combination

\[
c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0
\]

we must have \( c_1 = 0 \) since there is no other way to cancel \( \{t_1\} \). By induction on \( k \), the rest of the coefficients must also be zero.

Now we have all the ingredients to prove independence of the standard basis.
Proposition 3.7. The set \( \{ e_t \mid t \text{ is a standard } (\lambda, \mu)\text{-tableau} \} \) is linearly independent over \( K \).

Proof. By Corollary 3.5, \( \{ t \} \) is maximum in \( e_t \), and by hypothesis they are all distinct. Thus Lemma 3.6 applies.

We shall next show that standard \( (\lambda, \mu)\)-polytaboids span \( S^\lambda \mu \). In order to do this, we determine the Garnir element of \( t \) associated with \( e_t \) by using the following relations.

Lemma 3.8. Let \( t \) be a \( (\lambda, \mu)\)-tableau.

(i) For any two entries \( a \) and \( b \) in the same column of \( t \), \( (a, b) (-a, -b) \) generates a subgroup of order 2 of \( C_t \). Hence
\[
(e + (a, b) (-a, -b))e_t = 0 \quad \text{(alternacy relation)}
\]

(ii) For any entry \( a \) in the tableau \( t \)
\[
(e + \beta(a, -a))e_t = 0 \quad \text{(sign change relation)}
\]
where \( \beta = 1 \) if \( a \) appears in \( t_\lambda \) and \( \beta = -1 \) if \( a \) appears in \( t_\mu \).

Proof. The proof following immediately from (1.2) (i).

Remark 3.9. The previous lemma says that we can find the elements of \( O_a \) which remove any negative entries of \( t \) associated with \( e_t \) (sign change relation) and which make \( t \) column standard (alternacy relation), i.e. the tableau \( t \) associated with \( e_t \) may be reorganized so that no negative entries remain and all columns are standard.

Example 3.10. If we take
\[
t = \begin{pmatrix}
1 & 3 & 5 & 6 \\
4 & -2 & 7
\end{pmatrix}
\]
then
\[
e_t = -(2, -2)e_t = e^{1 \begin{pmatrix} 1 & 3 & 5 & 6 \end{pmatrix}^T} \begin{pmatrix} 4 & 2 & 7 \end{pmatrix}
\]
\[
= (2, 3) (-2, -3) e^{1 \begin{pmatrix} 1 & 3 & 5 & 6 \end{pmatrix}^T} \begin{pmatrix} 4 & 2 & 7 \end{pmatrix}
\]
\[
= e^{1 \begin{pmatrix} 1 & 2 & 5 & 6 \end{pmatrix}^T} \begin{pmatrix} 4 & 3 & 7 \end{pmatrix}
\]
(sign change relation)
alternacy relation
Now we want to find elements of the group algebra of $O_n$ which annihilate the given polytabloid $e_t$. Let $t$ be a $(\lambda, \mu)$-tableau associated with $e_t$ such that the entries of $t$ were reorganized by the sign change and alternacy relation. Suppose that $t$ has entries $t(i, j, k)$ and $t(i, j + 1, k)$, where $k = 1$ or $2$. Let $A$ be a set of entries in column $j$ of $t$ below and including $t(i, j, k)$, and let $B$ be the set of entries in column $j + 1$ of $t$ above and including $t(i, j + 1, k)$. Let $A = \{a_i : i \in I\}$ and $B = \{b_j : j \in J\}$, where $I$ and $J$ are index sets. Let $S_A, S_B, S_{A\cup B}$ be the subgroups of $O_n$. Let $\sigma_1, ..., \sigma_r$ be coset representatives for $S_A \times S_B$ in $S_{A\cup B}$, and let

$$S_{A\cup B} = \bigoplus_{j=1}^{r} \sigma_j(S_A \times S_B)$$ and $G_{A,B} = \sum_{j=1}^{r} (\text{sgn} \sigma_j) \sigma_j$.

The element $G_{A,B}$ is called a Garnir element of $t$ associated with $e_t$.

**Remark 3.11.** (i) If the elements of $A$ and $B$ belong to $t_{\lambda}$, then we assume that $S_A, S_B, S_{A\cup B}$ are the subgroups of $O_n$ generated by the elements in $A, B, A\cup B$ respectively. Let $C_{A\cup B} (\subset C_n)$ be the subgroup of $S_{A\cup B}$ generated by $\{w_i, w_j : i \in I, j \in J\}$. Since $C_{A\cup B}$ is also a subgroup of $S_A \times S_B$, all the coset representatives $\sigma_1, ..., \sigma_r$ are positive permutations.

(ii) If the elements of $A$ and $B$ belong to $t_{\mu}$, then we assume that $S_A, S_B, S_{A\cup B}$ are the subgroups of $O_n$ generated by positive permutations on $A, B, A\cup B$ respectively.

(iii) The coset representatives $\sigma_1, ..., \sigma_r$ are, of course, not unique, but for practical purposes note that we may take $\sigma_1, ..., \sigma_r$ so that $\sigma_1, ..., \sigma_r$ are all the tableaux which agree with $t$ except in the positions occupied by $A\cup B$, and whose entries increase vertically downwards in the positions occupied by $A\cup B$.

**Example 3.12.** Let $t$ be as in Example 3.10. Then $e_t = e^\{1, 2, 5, 6\}
\begin{align*}
&\begin{array}{ccc}
4 & 3 & 7
\end{array}
\end{align*}

Let $A = \{4\}$ and $B = \{2, 3\}$, and let $e$, $(34)$, $(234)$ be coset representatives for $S_A \times S_B$ in $S_{A\cup B}$. Then

$$G_{A,B} = e - (34) + (234).$$
Let \( H \) be any subset of \( O_n \). Define
\[
\overline{H} = \sum_{\sigma \in H} (\text{sgn } \sigma)\sigma
\]
and if \( H = \{\sigma\} \) then we write \( \overline{\sigma} = (\text{sgn } \sigma)\sigma \) for \( \overline{H} \).

**Lemma 3.13.** Let \( H \leq O_n \) be a subgroup.

(i) If the positive transposition \((a, b) (-a, -b) \in H\) then we can factor \( \overline{H} = k(e-(a, b) (-a, -b)) \), where \( k \in KO_n \).

(ii) If \( t \) is a \((\lambda, \mu)\)-tableau with \( a, b \) in the same row of \( t_\gamma \), where \( \gamma = \lambda \) or \( \mu \), and \((a, b) (-a, -b) \in H\), then \( \overline{H}(t) = 0 \).

**Proof.** (i) Consider the subgroup \( K = \{e, (a, b) (-a, -b)\} \) of \( H \). Select coset representatives \( \sigma_1, ..., \sigma_s \) for \( K \) in \( H \) and write \( H = \biguplus_{i=1}^s \sigma_i K \). But then,
\[
\overline{H} = \left( \sum_{i=1}^s \overline{\sigma_i} \right) (e - (a, b) (-a, -b)) ,
\]
as desired.

(ii) By hypothesis, \((a, b) (-a, -b) \{t\} = \{t\} \). Thus,
\[
\overline{H}(t) = k(e - (a, b) (-a, -b)) \{t\} = k(\{t\} - \{t\}) = 0 .
\]

**Proposition 3.14.** Let \( t, A, B \) be as in the definition of a Garnir element. If \(|\mu \cup B|\) is greater than the number of elements in column \( j \) of \( t \) then \( G_{A,B} c_t = 0 \).

**Proof.** Let
\[
S_A \times S_B = \sum_{\sigma \in S_A \times S_B} (\text{sgn } \sigma)\sigma \quad \text{and} \quad \overline{S}_{A \cup B} = \sum_{\sigma \in S_{A \cup B}} (\text{sgn } \sigma)\sigma .
\]
Consider any \( \sigma \in C_t \). If the elements of \( A \) and \( B \) are in the \( t_\lambda \), then by the hypothesis there must be positive integers \( a, b \in A \cup B \) such that \( c = \pm a \), \( d = \pm b \). Thus by Remark 3.11 (i) we have \((c, d) (-c, -d) \in S_{A \cup B} \). If the elements of \( A \) and \( B \) are in the \( t_\mu \), then by the hypothesis there must be positive integers \( a, b \in A \cup B \) such that \( a, b \) belong to the same row of \( \sigma t_\mu \). Therefore, \((a, b) (-a, -b) \in S_{A \cup B} \) by Remark 3.11 (ii).
In terms of the element of $A$ and $B$ appearing in the $t'_i$, where $\gamma = \lambda$ or $\mu$, $(c, d)$ or $(a, b)$ belong to the same group $S_{AB}$. But then $S_{AB} \{\sigma t\} = 0$ by Lemma 3.13. Since this is true for every $\sigma$ appearing in $\kappa$, we have $S_{AB} e_i = 0$.

Now $S_{AB} = \bigoplus_j \sigma_j(S_A \times S_B)$ so $S_{AB} = G_{AB} S_A \times S_B$ by Remark 3.11. Since $S_A \times S_B \subseteq C_i$ then $S_A \times S_B$ is a factor of $\kappa_i$ and $S_A \times S_B e_i = |S_A \times S_B| e_i$. Therefore,

$$0 = S_A \times S_B e_i = |S_A \times S_B| G_{AB} e_i$$

Thus, $G_{AB} e_i = 0$ when the base field is $Q$, and since all the tabloid coefficients here are integers the same holds over any field $K$.

Example 3.15. Referring to Example 3.10 and Example 3.12, we have

$$0 = G_{AB} e_{t'} = e_{t'} - e_{(34)k'} + e_{(234)k'},$$

where $t' = \begin{pmatrix} 1 & 2 & 5 & 6 \\ 4 & 3 & 7 \end{pmatrix}$, so

$$e_{t'} = e_{t'} - e_{(34)k'} + e_{(234)k'}.$$

We shall now use the Garnir relations, sign change relation and alternacy relation to prove that any polytabloid can be written as a linear combination of standard polytabloids. We have already shown how to do this in Example 3.15.

Theorem 3.16. The set $\{e_i \mid t$ is a standard $(\lambda, \mu)$-tableau$\}$ spans $S^{\lambda\mu}$.

Proof. First we write $[t]$ for the column equivalence class of $t$, that is

$$[t] = \{t' \mid t' = \sigma t$ for some $\sigma \in C_i\}.$$ 

The column equivalence classes are partially ordered in a way similar to the partial order in Definition 3.3 on the row equivalence classes.

Let $t$ be any $(\lambda, \mu)$-tableau associated with $e_i$. Then we may assume that $e_i$ may be written as a linear combination of column standard
polytabloids by Lemma 3.8. Therefore, because of Remark 3.9, we may always take \( t \) to have increasing columns and no negative entries.

Suppose that \( t \) is not standard. This means that \( t' \) is not standard, where \( \gamma = \lambda \) or \( \mu \). By induction, we may assume that \( e_s \) can be written as a linear combination of the polytabloids \( e_{t'} \), such that each \( t' \) is standard, where \( \gamma = \lambda \) or \( \mu \), when \([s] \triangleright [\tau]\) and prove the same result for \( e_t \) by considering \( t' \), where \( \gamma = \lambda \) or \( \mu \).

Now suppose that \( \gamma = \lambda \). Then there must be some adjacent pair of columns in the part \( t_\lambda \) of \( t \), say the \( j \)-th and \((j + 1)\)-th columns, with entries \( a_1 < a_2 < \ldots < a_p \) and \( b_1 < b_2 < \ldots < b_q \) with \( a_i > b_i \) for some \( i \). Thus we have the following situation in the part \( t_\lambda \) of \( t \):

\[
\begin{array}{c c c c c c c c c c c c c c c c}
& a_1 & b_1 & \wedge & & & \\
& a_2 & b_2 & \wedge & & & \\
& \vdots & \vdots & \wedge & & & \\
& a_i & \wedge & & & & \\
& a_{i+1} & \wedge & & & & \\
& \vdots & \vdots & \wedge & b_q & & \\
& a_p & \\
\end{array}
\]

Take \( A = \{a_1, a_{i+1}, \ldots, a_p\} \) and \( B = \{b_1, b_2, \ldots, b_i\} \), and consider the corresponding Garnir element \( G_{A,B} = \sum_\sigma (\text{sgn } \sigma)\sigma \). By Proposition 3.14, we have \( G_{A,B} e_t = 0 \) so that

\[
e_t = -\sum_{\sigma \in S_n} (\text{sgn } \sigma)e_{\sigma t}
\]
But $b_1 < b_2 < \ldots < b_i < a_i < \ldots < a_p$ implies that $[\sigma t] \triangleright [t]$ for $\sigma \neq e$ by the column analogue of the dominance lemma for tabloids (Lemma 3.4). Since $e_i = -\sum_{\alpha \in \lambda} (\text{sgn} \sigma)e\alpha$, the result follows from our induction hypothesis for the part $t\lambda$ of $t$.

If the part $t\mu$ of $t$ is also not standard then the result can be deduced when we repeat the above process by considering the part $t\mu$ of $t$.

ACKNOWLEDGEMENT

I would like to express my appreciation to Professor A. O. Morris who, as my supervisor, provided both assistance and encouragement.

REFERENCES


