ON THE (k+1)-DIMENSIONAL SPACE-LIKE RULED SURFACES IN THE MINKOWSKI SPACE $\mathbb{R}^n_1$

Murat TOSUN*, İsmail AYDEMİR**

* Department of Mathematics, Faculty of Sciences and Arts, Sakarya University, Sakarya/TURKEY
** Department of Mathematics, Faculty of Educations, Ondokuz Mayıs University, Samsun/TURKEY

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ABSTRACT

In this paper, space-like ruled surfaces in the Minkowski n-space are defined. Moreover, some results and theorems related with the Riemannian curvature $K$ and mean curvature vector $H$ of the $(k+1)$-dimensional space-like ruled surface are given.

1. INTRODUCTION

We shall assume throughout this paper all manifolds, maps, vector fields, etc... are differentiable of class $C^\infty$. Consider a general Semi-Riemannian submanifold $M$ of dimension $(k+1)$ of the Minkowski space $\mathbb{R}^n_1$ ($n \geq 3$). If $D$ (resp. $D$) is the Levi-Civita connection of $\mathbb{R}^n_1$ (resp. $M$) and if $X$ and $Y$ are tangent vector fields of $M$, then we find by decomposing $\bar{D}_X Y$ into a tangent and normal component

$$\bar{D}_X Y = D_X Y + V(X,Y) \quad (1.1)$$

$V(X,Y)$ is a normal vector filed on $M$ and is symmetric in $X$ and $Y$. A vector field $Z$ of $M$ $P$ is called an asymptotic vector field if $V(Z,Z) = 0$. A curve on $M$ is an asymptotic curve if its tangent vector field $T$ is an asymptotic vector field along the curve [1].

Let $\xi$ be a normal vector filed on $M$, then, by decomposing $\bar{D}_X \xi$ in a tangent and a normal component, we find that

$$\bar{D}_X \xi = -A_\xi(X) + D^\perp_X \xi \quad (1.2)$$

which determines, at each point, a self-adjoint linear map, where $D^\perp$ is a metric connection in the normal bundle $\chi^\perp(M)$. We use the same notation $A_\xi$ to show
the linear map and the matrix of the linear map. A normal vector field \( \xi \) on \( \chi(M) \) is called parallel on the normal bundle \( \chi^\perp(M) \) if \( D_X^\perp \xi = 0 \) for each vector field \( X \). A subbundle \( F \) of \( \chi^\perp(M) \) is said to be parallel in \( \chi^\perp(M) \) if for each vector field \( \eta \) of \( F \) and each vector field \( X \) of \( \chi^\perp(M) \), \( D_X^\perp \eta \) is again a vector field of \( F \), [2].

Suppose that \( X \) and \( Y \) are vector fields on \( \chi(M) \) while \( \xi \) is a normal vector field, then, if the standard metric tensor of \( R^n \) is denoted by \( \langle , \rangle \),

\[
\langle D_X Y, \xi \rangle = \langle V(X, Y) \xi \rangle = \langle A_\xi(X), Y \rangle. \tag{1.3}
\]

If \( \xi_1, \xi_2, \ldots, \xi_{n-k-1} \) constitute an orthonormal base field of the normal bundle \( \chi^\perp(M) \), then we put

\[
\langle V(X, Y) \xi_j \rangle = V_j(X, Y) \tag{1.4}
\]

or

\[
V(X, Y) = \sum_{j=1}^{n-m} V(X, Y) \xi_j.
\]

The mean curvature vector \( H \) of \( M \) at the point \( P \) is given by

\[
H = \sum_{j=1}^{n-k-1} \frac{\text{tr} A_{\xi_j}^\perp}{\text{vol} M} \xi_j. \tag{1.5}
\]

\( ||H|| \) shows the mean curvature. If \( H = 0 \) at each point \( P \) of \( M \), then \( M \) is said to be minimal, [1]. Let \( R^n \) be a Minkowski space in the Levi-Civita connection \( D \). The function,

\[
\bar{R} \colon \chi(R^n) \times \chi(R^n) \times \chi(R^n) \to \chi(R^n)
\]

given by

\[
\bar{R}(X, Y)Z = \overline{D_{[X,Y]}Z} - \overline{D_X D_Y Z} + \overline{D_Y D_X Z} \tag{1.6}
\]

is a \((1,3)\) tensor field on \( \chi(R^n) \) called the curvature tensor field of \( R^n \). If \( X, Y \in T^n(p) \) the linear operator

\[
R_{XY} \colon T^n(p) \to T^n(p)
\]

sending each \( Z \) to \( R_{XY}Z \) is called a curvature operator, [3]. The function
R: \( T_M(p) \times T_M(p) \times T_M(p) \times T_M(p) \rightarrow \mathbb{R} \)
given by
\[
R(X_1,X_2,X_3,X_4) = \langle X_1, R(X_3,X_4)X_2 \rangle
\] (1.7)
is a covariant tensor field of order 4 on \( \chi(M) \) called the Riemannian curvature tensor field of \( M \).

The function given by (1.7), at each point \( P \), is called the Riemannian curvature and we denote
\[
K(P) = \langle X,R(X,Y)Y \rangle. \quad (1.8)
\]

If \( V \) is the second fundamental form of Semi-Riemannian manifold \( M \), then we obtain
\[
\langle X,R(X,Y)Y \rangle = \langle V(X,Y),V(X,Y) \rangle - \langle V(X,X),V(Y,Y) \rangle. \quad (1.9)
\]

A two-dimensional subspace \( \pi \) of the tangent space \( T_M(p) \) is called a tangent plane to \( M \) at \( P \). For tangent vectors \( X_p,Y_p \in T_M(p) \) defined by
\[
K(X_p,Y_p) = \frac{\langle R(X_p,Y_p)X_p,Y_p \rangle}{\langle X_p,X_p \rangle \langle Y_p,Y_p \rangle - \langle X_p,Y_p \rangle^2} \quad (1.10)
\]
is called the sectional curvature of \( M \) at \( P \), [3].

2. (k+1)-DIMENSIONAL RULED SURFACE IN \( \mathbb{R}^n_1 \)

Let \( \{e_1(s), e_2(s), \ldots, e_k(s)\} \) be a system of orthonormal vector fields, which is defined for each point of a space-like curve \( \alpha \) in the \( n \)-dimensional Minkowski space \( \mathbb{R}^n_1 \). This system spans a \( k \)-dimensional subspace of the tangent space \( T_{\mathbb{R}_1^k}(\alpha(s)) \) at each point. This subspace is denoted by \( E_k(s) \), that is
\[
E_k(s) = \text{Sp}\{e_1(s), e_2(s), \ldots, e_k(s)\}.
\]

We get a \( (k+1) \)-dimensional surface in \( \mathbb{R}^n_1 \) if the subspace \( E_k(s) \) moves along the curve \( \alpha \). We call this space a \( (k+1) \)-dimensional generalized space-like ruled surface in the \( n \)-dimensional Minkowski Space \( \mathbb{R}^n_1 \). A parametrization of this ruled surface is
\[
\phi(s,u_1,\ldots,u_k) = \alpha(s) + \sum_{i=1}^{k} u_i e_i(s) \quad (2.1)
\]
Throughout this paper $E_k(s) = \text{Sp}(e_1(s), e_2(s), ..., e_k(s))$ denotes a subspace which is a space-like subspace, $\alpha$ is a space-like curve which is an orthogonal trajectory of the $k$-dimensional generating space $E_k(s)$ $(k \geq 1)$. We denote this ruled surface by $M$. If we take the partial derivate of $\phi$ we get

$$
\phi_s = \alpha(s) + \sum_{i=1}^{k} u_i e_i(s), \\
\phi_{u_i} = e_i(s), \ 1 \leq i \leq k.
$$

Throughout our paper we assume that the system

$$
\left\{ \alpha(s) + \sum_{i=1}^{k} u_i e_i(s), e_1, ..., e_k \right\}
$$

is linear independent.

Let $\{e_0, e_1, ..., e_k\}$ be an orthonormal base of $\chi(M)$ (i.e. $e_0$ is the unit tangent vector of the orthonogonal trajectories of the generating spaces). Suppose that timelike subspace $\{\xi_1, \xi_2, ..., \xi_{n-k+1}\}$ is an orthonormal base field of $\mathcal{T}_M^v(p)$. Then $\{e_0, e_1, ..., e_k, \xi_1, \xi_2, ..., \xi_{n-k+1}\}$ is a base field of $\mathcal{T}_q^v(p)$ at the point $P \in R^n_1$. Then we have

$$
\langle e_0, e_0 \rangle = 1, \langle e_i, e_0 \rangle = 0, \langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}, \langle \xi_j, \xi_j \rangle = \varepsilon_j = \pm 1. \quad (2.2)
$$

Then $M$ is said to be m-developable if

$$
\text{rank } [e_0, e_1, ..., e_k, D_{e_0} e_1, ..., D_{e_0} e_k] = 2k - m
$$

at each point $P \in M$. If $m = -1$, then the space-like ruled surface $M$ is called non-developable; if $m = k-1$, then $M$ is said to be total developable, [4].

Denote of $\overline{D}$ the Levi-Civita connection of the Minkowski space $R^n_1$. For the orthonormal base $\{e_1, ..., e_k\}$ of the generating space $E_k(s)$, we observe that

$$
\overline{D}_{e_i} e_j = 0, \ 1 \leq i, j \leq k.
$$

Hence, if $V$ denotes the second fundamental form of $R^n_1$, we must have

$$
V(e_j, e_j) = 0, \ 1 \leq i, j \leq k. \quad (2.4)
$$
Let $X = \sum_{i=1}^{k} a_i e_i + ae_0$ and $Y = \sum_{i=1}^{k} b_i e_i + be_0$ be two vector fields of $\chi(M)$. So we find that

$$V(X,Y) = \sum_{i=1}^{k} (a_i b + b_i a) V(e_0 e_i) + ab V(e_0 e_0).$$

The normal subbundle of $\chi^1(M)$ spanned by the normal fields $V(e_0 e_i)$, $1 \leq i \leq k$ is denoted by $F$.

**Theorem 2.1.** $M$ is m-developable iff the normal subbundle $F$ is $(k-m-1)$-dimensional.

**Proof.** Suppose that we have (2.3). Because of (1.1) we can write

$$\bar{D}_{e_0} e_i = D_{e_0} e_i + V(e_0 e_i), \quad 1 \leq i \leq k.$$ 

But $D_{e_0} e_i$ is a linear combination of the vector fields $\{ e_0, e_1, \ldots, e_k \}$ and so we may replace the fields $\bar{D}_{e_0} e_i$ by $V(e_0 e_i)$ in (2.3). Now, the tangent space spanned by $e_0, e_1, \ldots, e_k$ is at each point normal to $F$ and thus we find $k+1\text{-dim } F = 2k-m$ or $\dim F = k-m-1$, which completes the proof of the theorem.

From (2.2) we observe that $\bar{D}_{e_0} e_0 \perp e_0$ and $\bar{D}_{e_i} e_0 \perp e_i$. This means that $\bar{D}_{e_0} e_i$ is a normal vector field of

$$\bar{D}_{e_i} e_0 = V(e_i, e_0), \quad 1 \leq i \leq k.$$ 

Suppose that $\{ \xi_1, \xi_2, \ldots, \xi_{n-k-1} \}$ is an orthonormal base field of the normal bundle $\chi^1(M)$, then we have the following Weingarten equations

$$\bar{D}_{e_0} \xi_j = a_{00}^j e_0 + \sum_{r=1}^{k} a_{or}^j e_r + \sum_{s=1}^{n-k-1} b_{0s}^j \xi_s, \quad 1 \leq j \leq n-k-1,$$

$$\bar{D}_{e_i} \xi_j = a_{ih}^j e_0 + \sum_{r=1}^{k} a_{ih}^j e_r + \sum_{s=1}^{n-k-1} b_{is}^j \xi_s,$$

$$\bar{D}_{e_k} \xi_j = a_{k0}^j e_0 + \sum_{r=1}^{k} a_{kr}^j e_r + \sum_{s=1}^{n-k-1} b_{ks}^j \xi_s.$$ 

These equation together with (2.4) and (1.3) yield

$$a_{or}^j = a_{ro}^j \quad (2.8)$$

$$a_{i}^j = 0 \quad 1 \leq j \leq n-k-1, \quad 1 \leq i, r \leq k.$$ 

So the matrix of $A_{ij}$ has the form
\[
A_{\xi_j} = \begin{bmatrix}
\tilde{a}_{\xi_0}^j & \tilde{a}_{\xi_1}^j & \cdots & \tilde{a}_{\xi_k}^j \\
\tilde{a}_{\xi_0}^j & 0 & \cdots & 0 \\
\tilde{a}_{\xi_1}^j & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
\tilde{a}_{\xi_k}^j & 0 & \cdots & 0 
\end{bmatrix}
\] (2.9)

and this means \( \det A_{\xi_j} = 0 \) if \( k \geq 2 \), from which we have:

**Corollary 2.2.** If \( k \geq 2 \), then the Lipschitz-Killing curvature of \( M \) is zero at each point in each normal direction.

**Corollary 2.3.** The matrix \( A_{\xi_j} \) of the shape operator of \( M \) is of the form (2.9) and is symmetric.

Because of the equations (2.7), we get
\[
a_{\xi_i}^j = \langle \overline{D}_{\xi_i} \xi_j, e_0 \rangle = -\langle \xi_i, \overline{D}_{\xi_i} e_0 \rangle
\] (2.10)

and from (2.6) together with (2.10) we receive
\[
\overline{D}_{\xi_i} e_0 = V(e_i, e_0) + \sum_{j=1}^{n-k-1} \epsilon_j \langle \xi_j, \overline{D}_{\xi_i} e_0 \rangle \xi_j = -\sum_{j=1}^{n-k-1} \epsilon_j a_{\xi_i}^j \xi_j .
\] (2.11)

**Theorem 2.4.** Let \( M \) be a \((k+1)\)-dimensional space-like ruled surface of \( \mathbb{R}^n_1 \). Then the Riemannian curvature of \( M \) in the two-dimensional direction spanned by \( e_i \) and \( e_0 \) is given by
\[
\mathbf{K}(e_i, e_0) = \langle \overline{D}_{e_0} \overline{D}_{e_i} e_0, e_0 \rangle, \quad 1 \leq i \leq k .
\]

**Proof:** Let \( R \) be the Riemannian curvature tensor field of \( M \). From (1.10) and (2.2), we find
\[
\mathbf{K}(e_i, e_0) = \langle R(e_i, e_0) e_i, e_0 \rangle .
\] (2.12)

If we connect (2.12) with (1.9) and (2.4), then we get
\[
\mathbf{K}(e_i, e_0) = \langle V(e_i, e_0), V(e_i, e_0) \rangle
\]
or
\[
\mathbf{K}(e_i, e_0) = \langle \overline{D}_{e_0} \overline{D}_{e_i} e_0, e_0 \rangle .
\] (2.13)

From (2.11) and (2.13) we receive the following corollary.

**Corollary 2.5.** The Riemannian curvature of \( M \) in the two-dimensional direction spanned by \( e_i \) and \( e_0 \) can be written with the entries of the Matrix \( A_{\xi_j} \) as follows
\[ K(e_i e_0) = \sum_{j=1}^{n-k-1} \xi_j (a_u^i)^2 , \quad 1 \leq i \leq k , \quad \xi_j = \langle \xi_j, \xi_j \rangle = \pm 1 . \tag{2.14} \]

It is easy to see that (1.10) and (2.4) gives

\[ K(e_i e_j) = 0 , \quad 1 \leq i , j \leq k . \tag{2.15} \]

**Theorem 2.6.** Let \( M \) be a \((k+1)\)-dimensional space-like ruled surface in \( \mathbb{R}^n_1 \) and \( e_0 \) be the tangent vector field of the base curve of \( M \). The mean curvature is

\[ H = \xi_j \frac{V(e_0 e_0)}{k + 1} , \quad \xi_j = \langle \xi_j, \xi_j \rangle = \pm 1 . \]

**Proof:** From (1.5) we known that

\[ H = \sum_{j=1}^{n-k-1} \frac{\text{tr} A_{\xi_j}}{k + 1} \xi_j . \tag{2.16} \]

Using (1.4), we can write

\[ V(e_0 e_0) = \sum_{j=1}^{n-k-1} \xi_j (D e_0 e_0 \xi_j)^2 , \quad \xi_j = \langle \xi_j, \xi_j \rangle = \mp 1 \]

Because of the last equation and equation (2.7), we get

\[ V(e_0 e_0) = -\sum_{j=1}^{n-k-1} \xi_j \left( a_u^j \right) \xi_j \]

For the matrix \( A_{\xi_j} \) given (2.9) we find

\[ \text{tr} A_{\xi_j} = -a_{00}^j . \tag{2.18} \]

If we substitute (2.17) and (2.18) in (2.16), we observe that

\[ H = \xi_j \frac{V(e_0 e_0)}{k + 1} , \quad \xi_j = \langle \xi_j, \xi_j \rangle = \pm 1 \]

From Theorem 2.6 we have immediately:

**Corollary 2.7.** The space-like ruled surface \( M \) is minimal iff each orthogonal trajectory of the generating spaces is an asymptotic line of \( M \).

**Theorem 2.8.** If the \((k+1)\)-dimensional m-developable space-like ruled surface \( M \) is minimal, then \( M \) is necessarily a submanifold of an \( \mathbb{R}^{2m}_1 \).

**Proof:** Because of Theorem 2.1, we already know that the codimension of \( M \) is at least \( k-m-1 \) we have two cases:

1) First, suppose that the normal subbundle \( F \) is zero-dimensional. Thus
\( V(e_0,e_i) = 0, \ 1 \leq i \leq k. \)

Because of the second fundamental form \( V \) is symmetric, we find
\( V(e_i,e_0) = 0, \ 1 \leq i \leq k. \)

If we substitute \( V(e_0,e_0) = 0 \) and \( V(e_i,e_0) = 0, \ 1 \leq i \leq k \) in (2.5), we get
\( V(X,Y) = 0. \)

This says that the space-like ruled surface \( M \) must be totally geodesic, i.e. \( M \) is part of a \((k+1)\)-dimensional linear space.

2) Next assume that the normal subbundle \( F \) is not zero. Consider an orthonormal base field \( \xi_1, \xi_2, \ldots, \xi_{n-k-1} \) of \( \chi^1(M) \) such that \( \xi_1, \xi_2, \ldots, \xi_{k+m-1} \) is a base field of the normal subbundle \( F \). Consider the equations (2.7) in this case. Since \( \langle V(e_i,e_0), \xi_j \rangle = -\delta_{bi}^j, \ 1 \leq i \leq k, \ 1 \leq j \leq n-k-1 \) we have immediately
\[
\delta_{bi}^j = 0, \quad 1 \leq i \leq k, \quad k-m \leq j \leq n-k-1. \tag{2.19}
\]

But \( H = 0 \) and hence \( \text{tr} \ A_{\xi_j} = 0, \ 1 \leq j \leq n-k-1 \) and so we get
\[
A_{\xi_1,\xi_2} = \ldots = A_{\xi_{n-k-1},\xi_{n-k-1}} = 0. \tag{2.20}
\]

If we set \( V(X,Y) = \sum_{j=1} V_j(X,Y) \xi_j \) for each two vector fields \( X \) and \( Y \) of \( M \), then we find
\[
V_{k+m}(X,Y) = \ldots = V_{n-k-1}(X,Y) = 0. \tag{2.21}
\]

If \( \overline{R} \) is the curvature tensor of \( R^n_1 \) and if \( X, Y, Z \) are vector fields of \( \chi(M) \), then the Codazzi equation says
\[
\left( \overline{R}(X,Y)Z \right)^{\perp} = \sum_{j=1}^{n-k-1} \left( \langle D_Y V_j, (XZ) \rangle - \langle D_X V_j, (YZ) \rangle \xi_j \right) + \sum_{j=1}^{n-k-1} V_j(XZ)D_Y \xi_j - \sum_{j=1}^{n-k-1} V_j(YZ)D_X \xi_j. \tag{2.22}
\]

Put
\[
D^\perp_{\xi_j} \xi_k = \sum_{n=1}^{n+k} C^h_{i,k} \xi_n + \sum_{r=k+m}^{n+k} C^r_{i,k} \xi_r, \quad 1 \leq \xi \leq k-m-1, \ 1 \leq i \leq k. \tag{2.23}
\]

Then, from (2.21) and (2.22), we have
\[
\left( \overline{R} (e_i e_0) e_s \right)_s = \sum_{\ell = 1}^{k-m-1} \left\{ \left( D_{e_0} V_{\ell} \right) (e_i e_0) - \left( D_{e_i} V_{\ell} \right) (e_0 e_s) \right\} \xi_{\ell} \\
- \sum_{\ell = 1}^{k-m-1} V_{\ell} (e_0 e_s) D_{e_0} \xi_{\ell} + \sum_{\ell = 1}^{k-m-1} V_{\ell} (e_0 e_s) D_{e_i} \xi_{\ell} = 0, \quad 1 \leq i, s \leq k. \quad (2.24)
\]

But \( V (e_i e_s) = 0 \), \( 1 \leq i, s \leq k \) and so we find from (2.23) and (2.24)
\[
\sum_{\ell = 1}^{k-m-1} C_{i\ell}^{\xi} V_{\ell} (e_0 e_s) = 0, \quad 1 \leq i, s \leq k, \quad k-m \leq r \leq n-k-1. \quad (2.25)
\]

Now, fix in this expression \( i \) and \( r \) and let \( s \) be variable, then we find a system of \( k \) homogeneous linear equations with \( k-m-1 \) unknowns \( C_{i\ell}^{\xi} \).

The matrix of this system is
\[
[V_{\ell} (e_0 e_s)], \quad 1 \leq \ell \leq k-m-1, \quad 1 \leq s \leq k.
\]

and its rank is at each point of \( M \) equal to \( k-m-1 \) because space-like ruled surface \( M \) is \( m \)-developable. So, it is easy to see that (2.25) gives
\[
C_{i\ell}^{\xi} = 0, \quad 1 \leq i \leq k, \quad 1 \leq \ell \leq k-m-1, \quad k-m \leq r \leq n-k-1. \quad (2.26)
\]

We also have
\[
\left( \overline{R} (e_0 e_i) e_0 \right)_s = \sum_{\ell = 1}^{k-m-1} \left\{ \left( D_{e_i} V_{\ell} \right) (e_0 e_0) - \left( D_{e_0} V_{\ell} \right) (e_i e_0) \right\} \xi_{\ell} \\
+ \sum_{\ell = 1}^{k-m-1} V_{\ell} (e_0 e_0) D_{e_0} \xi_{\ell} - \sum_{\ell = 1}^{k-m-1} V_{\ell} (e_i e_0) D_{e_0} \xi_{\ell} = 0. \quad (2.27)
\]

But \( V (e_0 e_0) = 0 \), and if we put
\[
D_{e_0} \xi_{\ell} = \sum_{h=1}^{k-m-1} C_{h\ell}^{\xi} \xi_{h} + \sum_{r=k-m}^{n-k-1} C_{r\ell}^{\xi} \xi_{r}, \quad 1 \leq \ell \leq k-m-1
\]

we find from (2.27)
\[
\sum_{\ell = 1}^{k-m-1} C_{i\ell}^{\xi} V_{\ell} (e_i e_0), \quad 1 \leq i \leq k, \quad k-m \leq r \leq n-k-1.
\]

This gives analogously
\[
C_{i\ell}^{\xi} = 0, \quad 1 \leq \ell \leq k-m-1, \quad k-m \leq r \leq n-k-1. \quad (2.28)
\]

Now, equation (2.26) together with equation (2.28) says that for each unit normal field \( \eta \) in \( F \) and for each vector field \( X \) of \( M, D_{x}^{+} \eta \) has no component in the complementary subbundle \( F^{\perp} \) i.e. the normal subbundle \( F \) is parallel. If we identify all the tangent spaces of \( R_{1}^{n} \) with \( R_{1}^{n} \) itself, then, since \( F \) is parallel and because of equation (2.20), we see that the \((2k-m)\)-dimensional subspaces of \( R_{1}^{n} \) spanned at each point of \( M \) by the
tangent space and the normal space F, are independent of the choice of the point P of M, which was to be proved.

**Theorem 2.9.** If the mean curvature vector $H \neq 0$ of the $(k+1)$-dimensional m-developable space-like ruled surface M is at each point of M a vector of the normal subbundle F, then M is necessarily a submanifold of an $R_{2km}^1$.

**Proof:** Take again an orthonormal base field $\xi_1, \xi_2, \ldots, \xi_{n-k-1}$ such that $\xi_1, \xi_2, \ldots, \xi_{k-m-1}$ is a base field of F. Then, since $\varepsilon_j^{(k+1)}H = V(e_0,e_0) \in F$ we have again

$$V_{k-m}(X,Y) = \ldots = V_{n-k-1}(X,Y) = 0$$

for each two vector fields X and Y of $\chi(M)$.

Next, if we have (2.23), then we find from (2.24) again (2.26). Moreover, since the vector fields $D_{\xi_1}^{\xi_{k-m}}, 1 \leq i \leq k, 1 \leq \ell \leq k-m-1$ have no components in the complementary subbundle $F^\perp$, we find because of (2.27) again (2.28) and this completes the proof.

**REFERENCES**


