THE REPRESENTATION OF SERIES-PARALLEL-ORDERED SETS

ANDREAS TIEFENBACH

Department of Mathematics, Middle East Technical University, TURKEY

(Received March 31, 1997; Accepted June 17, 1997)

ABSTRACT

In [6] it is shown, that weak orders, a subclass of series parallel posets, are represented by bands. In this paper a representation of series parallel posets is given and it is shown how all weak ordered finite bands can be constructed. We first want to give a construction of CDC's as a set of special n-tupels of natural numbers. After this we assign to each of these tupels a rectangular band and show how weak ordered bands can be thus constructed. Moreover all weak ordered bands are constructed in this way.

1. INTRODUCTION

In 1986 Mitsch [3] showed that to any semigroup S a natural partial order can be defined by

\[ a \leq b \text{ if } a = bx = yb = ax \text{ for some } x, y \in S^1. \]

This order is an extension of the natural partial order on idempotent elements. In [4] Neggers showed that posets and poset homomorphisms form a category which is equivalent to the category of pogroupoids. This idea was carried on in [2], where it is shown that a pogroupoid of a weak order is a semigroup.

Therefore the following question seemed natural: When is a poset a natural poset of a semigroup? We also say that a semigroup represents a poset if the natural poset of the semigroup is isomorphic with the given poset. Hence we can reformulate the above question: When is a poset represented by a semigroup? Some classes are known to be represented, which we want to introduce in this paper.

We use \( a < b \) to express \( a \leq b \) but \( a \neq b \). And if \( a \preceq b \) and \( b \preceq a \) then we write \( a \parallel b \). A poset is called a weak order if \( \parallel \) defines an equivalence relation.
We need the equivalence $J$, one of Green's relation, given by

$$a \ J \ b \text{ if } S^1aS^1 = S^1bS^1.$$  

On the bands we have $a \ J \ b$ if $a = aba$ and $b = bab$ [1]. $J$ is used in theorem 4.1 and in the proof of theorem 4.2 and theorem 4.3.

2. SERIES PARALLEL POSETS

A partial ordered set is called series parallel if it can be constructed from singeltons using the operations of disjoint sum, denoted by '$+$', and linear sums, denoted by '$\oplus$'. For example trees are seriesparallel as well as weak orders. A known result is:

Theorem 2.1. (Series-parallel-N-Theorem) [5] [9] A finite ordered set is series parallel if and only if it contains no subset isomorphic to $N$.

To prove the main theorem we need results found in [7]. Special bands are used there, which are defined as follows:

Definition 2.2. A band (respectively a semigroup) is called a RZ band (respectively a RZ semigroup) if its set of minimal elements form a rightzero semigroup.

Theorem 2.3. [7] Let $\mathcal{Q}_i$, $i = 1, 2$ be orders which are represented by RZ bands, then $\mathcal{Q}_1 + \mathcal{Q}_2$ is represented by $(B_1 \cup B_2, \ast)$. The multiplication is given by

$$x \ast y = \begin{cases}  
m_{i,j} & \text{if } x \notin B_i, y \in B_i \\
xy & \text{else} \end{cases}$$

where $m_{i,j}$ is a fixed minimal element in $B_i$.

Lemma 2.4. [7] Let $\mathcal{Q}_i$, $i = 1, 2$ two orders which are represented by bands $B_i$, $i = 1, 2$ then $\mathcal{Q}_1 \oplus \mathcal{Q}_2$ is represented by $(B_1 \cup B_2, \ast)$ where

$$x \ast y = \begin{cases}  
x & \text{if } x \in B_1, y \in B_2 \\
y & \text{if } x \in B_2, y \in B_1 \\
xy & \text{else} \end{cases}$$
These results show that if two posets are represented by RZ bands, then also their disjoint and linear sum is represented by a RZ band. Consequently:

**Theorem 2.5.** Any series parallel order \( \square \) is represented by a RZ band \( B \).

**Proof.** Let \( a \) be the defining expression of \( \square \). If \( \square \) is not a singelton the expression consists of two smaller subexpressions, that are connected by either '+' or '\( \oplus \)'. Since we showed that a cardinal sum as well as a linear sum of orders, which are represented by RZ bands are again represented by RZ bands and obviously the singelton is represented by a RZ band, an easy induction argument on the length of the expression \( a \) completes the proof.

**Example 2.6.** Let \( a \) be the defining expression of a series parallel ordered set \( \square \), given by

\[
a = (1 \oplus (((1+1) \oplus (1 + 1)) + (1 \oplus (1+1) \oplus 1))) \oplus (1+1)
\]

To distinguish between the elements we rewrite \( a \) as:

\[
a = (e_1 \oplus (((c_2+c_3) \oplus (c_4+c_5)) + (c_6 \oplus (c_7+c_8) \oplus c_9))) \oplus (c_{10}+c_{11})
\]

The Hasse diagram of this order is:

```
Expression of the form \( e_i + e_j \) are represented by bands with the following multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>( e_i )</th>
<th>( e_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_i )</td>
<td>( e_i )</td>
<td>( e_j )</td>
</tr>
<tr>
<td>( e_j )</td>
<td>( e_i )</td>
<td>( e_j )</td>
</tr>
</tbody>
</table>
```
Using this for \( e_{10} \) and \( e_{11} \) we get the table for \( P_3 \). The tables for \( P_1 \) and \( P_2 \) are combinations of this table as described above. We get

\[
\begin{array}{c|ccccc}
* & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\
e_2 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\
e_3 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\
e_4 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\
\end{array}
\]

Finally the complete table is given by:

\[
\begin{array}{c|ccccc|ccccc}
* & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \\
\hline
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \\
e_2 & e_1 & e_1 & e_1 & e_1 & e_1 & e_1 & e_1 & e_1 & e_1 & e_1 \\
e_3 & e_1 & e_2 & e_3 & e_2 & e_3 & e_3 & e_3 & e_3 & e_3 & e_3 \\
e_4 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_6 & e_6 & e_6 & e_6 \\
e_5 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_6 & e_6 & e_6 & e_6 \\
e_6 & e_1 & e_2 & e_3 & e_2 & e_2 & e_7 & e_7 & e_7 & e_7 & e_7 \\
e_7 & e_1 & e_2 & e_3 & e_2 & e_2 & e_7 & e_7 & e_7 & e_7 & e_7 \\
e_8 & e_1 & e_2 & e_3 & e_2 & e_2 & e_7 & e_7 & e_7 & e_7 & e_7 \\
e_9 & e_1 & e_2 & e_3 & e_2 & e_2 & e_7 & e_7 & e_7 & e_7 & e_7 \\
e_{10} & e_1 & e_2 & e_3 & e_4 & e_5 & e_7 & e_7 & e_7 & e_7 & e_7 \\
e_{11} & e_1 & e_2 & e_3 & e_4 & e_5 & e_7 & e_7 & e_7 & e_7 & e_7 \\
\end{array}
\]

3. CROWN DIAMONDS CHAINS

We recall some definitions and a result found in [6]. Some finite semilattice have a special form and are obviously weakly ordered.

**Definition 3.1.** A semilattice \( Y \) of the form

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \</c> \end{array}
\]

A Crown

is called a crown.
**Definition 3.2.** A semilattice $Y$ of the form

![Diagram of a Diamond]

A Diamond

is called a diamond.

Let $\mathcal{Q}_i$, $i = 1, 2$ be two posets such that has $\mathcal{Q}_i$ a greatest element $g$ and $\mathcal{Q}_2$ has a smallest element $s$. Then the glue-linear sum of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is defined to be

$$\mathcal{Q}_1 \oplus (\mathcal{Q}_2 \setminus \{s\}) = (\mathcal{Q}_1 \setminus \{g\}) \oplus \mathcal{Q}_2.$$

**Definition 3.3.** A semilattice $Y$ is called a crown-diamond-chain if it is glue-linear sum of chains, diamonds and a crown as last summand or a glue-linear sum of chains and diamonds.

![Diagram of a Crown-Diamond-Chain]

A Crown-Diamond-Chain

The following theorem gives a description of finite weakly ordered semilattices:

**Theorem 3.4.** [6] A finite semilattice $Y$ is weak-ordered if and only if it is a crown-diamond-chain.
Since a CDC is weak ordered it is also a series parallel order. The defining expression of a CDC has the following obvious properties:

1. it doesn’t start with \((1 + \ldots + 1)\) and

2. it contains no subpart \(\left(\frac{1 + \ldots + 1}{k_1}\right) \oplus \left(\frac{1 + \ldots + 1}{k_2}\right), \ k_1, k_2 > 1\).

Consequently and CDC can be represented by a sequence

\[ c = (x_1, x_2, \ldots, x_n) \text{ with } \begin{cases} x_i > 0 & \text{if } i \text{ is odd} \\ x_i > 1 & \text{if } i \text{ is even.} \end{cases} \]

Such a sequence is transformed into a defining expression as follows:

\[ a = \left(\frac{1 \oplus \ldots \oplus 1}{x_1}\right) \oplus \left(\frac{1 + \ldots + 1}{x_2}\right) \oplus \left(\frac{1 \oplus \ldots \oplus 1}{x_3}\right) \oplus \ldots \]

From this it should be clear how to receive a sequence from a defining expression.

But these sequences do not only describe the general structure of a CDC. With its help a set of n-tuppels can be given, such that each element of the given CDC corresponds to one of these tuppels and this set can be endowed with a multiplication, that yields a CDC structure.

**Definition 3.5.** Let \( c = (x_1, \ldots, x_n) \) be a sequence that describes a CDC, then \( \langle c \rangle \) denotes the set of all n-tuppels \( (t_1, \ldots, t_n) \) such that \( 0 \leq t_i \leq x_i \) and if \( t_i < x_i \) than \( t_{i+1} = 0 \) and max \( (t_i : i = 1, \ldots, n) > 0 \).

**Lemma 3.6.** Let \( c = (x_1, \ldots, x_n) \) then \( |\langle c \rangle| = \sum_{i=1}^{n} x_i \).

**Proof.** We prove this by induction. Let the length of \( c \) be one, then the result follows immediatly. Now we assume that for sequences \( c \) with length \( k \) the result is true. If we take a sequence with length \( k+1 \), then we have \( \sum_{i=1}^{n} x_i \) elements with \( t_{k+1} = 0 \). Adding the \( x_{k+1} \) cases for \( t_k > 0 \) we get the result. Note that if \( t_{k+1} > 0 \) then \( t_i = x_i \) for \( i \leq k \).

Now that we saw that \( \langle c \rangle \) has exactly the same number than the CDC which is defined by \( c \), we will give a multiplication on \( \langle c \rangle \) such that it becomes a CDC with the same defining expression, that means \( \langle c \rangle \) represents the given CDC and at the same time we saw, that all CDC can be constructed as a set \( \langle c \rangle \) for some, one, sequence \( c \).
Theorem 3.7. Let \( c = (x_1, \ldots, x_n) \). If we define on \( \langle c \rangle \) the following multiplication:

\[(s_1, \ldots, s_n) \ast (t_1, \ldots, t_n) = ([s_1, t_1], \ldots, [s_n, t_n])\]

where

\[ [s_i, t_i] = \begin{cases} 0 & \text{if } i \text{ is even and } s_i \neq t_i \\ \min(s_i, t_i) & \text{else} \end{cases} \]

then \( \langle (c), \ast \rangle \) becomes a CDC.

Proof. The given multiplication is idempotent and commutative. We have to show that it is associative and closed. It suffices to show that \( [\_, \_] \) is associative on the components. We observe the case where \( i = 2k \) and \( i \neq 2k \).

\[ i \neq 2k \] \[ ([a_i, b_i], d_i) = \min(\min(a_i, b_i), d_i) = \min(a_i, b_i, d_i) = [a_i, [b_i, d_i]] \]

Now we turn to the even components:

\[ [a_i, b_i], d_i] := \begin{cases} [a_i, d_i] & \text{if } a_i = b_i \\ [0, d_i] = 0 & \text{if } a_i \neq b_i \end{cases} \]

\[ [a_i, b_i], d_i] := \begin{cases} [a_i, d_i] & \text{if } b_i = d_i \\ [a_i, 0] = 0 & \text{if } b_i \neq d_i \end{cases} \]

To check whether these products are equal, we have to consider two cases: \( a_i = d_i \) and \( a_i \neq d_i \).

If \( a_i = d_i \) and \( a_i = b_i \) then \( b_i = d_i \) too and \( [a_i, b_i], d_i] = a_i = [a_i, [b_i, d_i]] \). If \( a_i = d_i \) and \( a_i \neq b_i \) then \( d_i \neq b_i \) too and we have \( [a_i, b_i], d_i] = [0, d_i] = 0 \) \( [a_i, 0] = [a_i, [b_i, d_i]] \). Now we look after \( a_i \neq d_i \). In this case all of the above outcomes are 0. Consequently the multiplication \( \ast \) is associative since:

\[(a \ast b) \ast d = ([a_1, b_1], d_1], \ldots, [a_n, b_n], d_n] = ([a_1, b_1], d_1], \ldots, [a_n, b_n], d_n] = a \ast (b \ast d)\]

It remains to show that the multiplication is closed. If \( a = b \) then we know that \( a \ast b = a \ast a = a \). Let \( a \neq b \), then there is an index \( i \), minimal, such that \( a_i \neq b_i \), say \( a_i < b_i \). Consequently \( a_i < x_i \) and \( a_{i+1} = \ldots = x_{i+1} = \ldots = a_n = b_n = d_n = 0 \).
Moreover $a_k = 0$ when $k > i$. All components $d_k = [a_k, b_k]$ of $a*b$ with $k > i$ are 0 and the components $d_k$ with $k < i$ are $a_k = b_k$. If $i$ is even then $d_i = 0$ and $a*b \in \langle c \rangle$. If $i$ is odd then $d_i = \min(a_i, b_i) = a_i$ and $a*b$ is in $\langle c \rangle$.

This semilattice is obviously a CDC.

4. WEAK ORDERED BANDS

Now we assign to each element of a CDC $((c), *)$ a rectangular band $RB_a$, $a \in \langle c \rangle$. The rectangular bands $RB_a$ are arbitrary except for the elements $a = (x_1, x_2, \ldots, x_{2k+1}, 0, \ldots, 0)$. For these elements $RB_a$ consists of only one element, say $x_a$. On the set

$$W := \{(a, x) : a \in \langle c \rangle, x \in RB_a\}$$

we define the following multiplication.

$$(a, x) \circ (b, y) = (a * b \theta(a, b, x, y))$$

where $\theta(a, b, x, y)$ is defined by

$$\theta(a, b, x, y) := \begin{cases} x & \text{if } a = a * b, a \neq b \\ y & \text{if } b = a * b, a \neq b \\ xy & \text{if } a = b \\ x_{a*b} & \text{else} \end{cases}$$

Here $x_{a*b}$ is the only element in $RB_{a*b}$.

We need the following

**Theorem 4.1.** [6] A finite band $B$ is weak ordered if and only if the following properties hold:

1. $B/J$ is a crown-diamond-chain and

2. $a < b \iff aJ < bJ$

to show that

**Theorem 4.2.** $(W, \circ)$ is a weak ordered band.
Proof. The given multiplication is obviously closed and it is easy to see, that it is idempotent, since

\[(a,x) \circ (a,x) = (a \ast a, \theta(a, a, x, x)) = (a,x)\]

Now we show that \(\circ\) is a associative.

\[
\begin{align*}
((s_1,x_1) \circ (s_2,x_2)) & \circ (s_3,x_3) = \\
= (s_1 \ast s_2, \theta(s_1, s_2, x_1, x_2)) & \circ (s_3,x_3) = \\
= ((s_1 \ast s_2) \ast s_3, \theta(s_1 \ast s_2, s_3, x_1, x_2)) \circ (s_3,x_3) \\
& = (s_1 \ast s_2 \ast s_3, \theta(s_1, s_2 \ast s_3, x_1, x_2)) \\
\end{align*}
\]

So the multiplication is associative if (1) = (2). The left side, this is (1) depends on \(\theta(s_1, s_2, x_1, x_2)\) hence we get

L1 \(\theta(s_1, s_2, x_1, x_2)\) if \(s_1 < s_2\)

L2 \(\theta(s_2, s_3, x_2, x_3)\) if \(s_2 < s_1\)

L3 \(\theta(s_1, s_2, x_1, x_2, x_3)\) if \(s_2 = s_3\)

L4 \(\theta(s_1 \ast s_2, s_3, x_1, x_2, x_3)\) if \(s_1 \parallel s_2\)

The right side depend on \(\theta(s_1, s_2, x_1, x_2, x_3)\) and we get

R1 \(\theta(s_1, s_2, x_1, x_2)\) if \(s_2 < s_3\)

R2 \(\theta(s_1, s_3, x_1, x_3)\) if \(s_3 < s_2\)

R3 \(\theta(s_1, s_2, x_1, x_2, x_3)\) if \(s_2 = s_3\)

R4 \(\theta(s_1, s_2 \ast s_3, x_1, x_2, x_3)\) if \(s_2 \parallel s_3\)

To check associativity we have to show that \(L_i = R_j \forall i,j\).

L1R1 \(s_1 < s_2\) and \(s_2 < s_3\) hence \(s_1 < s_3\) and \(L1 = x_1 = R1\).

L1R2 L1 equals R2 indeed.

L1R3 \(s_1 < s_2\) and \(s_2 = s_3\) yields \(s_1 < s_3\) hence \(L1 = x_1 = R3\).
L1R4 $s_1 < s_2$ and $s_2 \parallel s_3$ yields $s_1 < s_3$ since $\langle c \rangle$ is a CDC, hence $L1 = x_1 = R4$.

L2R1 $s_2 < s_1$ used in R1 gives $x_2$ and $s_2 < s_3$ yields $x_2$ in L2.

L2R2 $s_2 < s_1$ and $s_3 < s_2$ yields $s_3 < s_1$ hence $L2 = x_3 = R2$.

L2R3 $s_2 < s_1$ and $s_3 = s_2$. Hence $s_3 < s_1$ and $L2 = x_2x_3 = R3$.

L2R4 Since $s_2 \parallel s_3$ and because $\langle c \rangle$ is a CDC we have also $s_3 < s_1$. Consequently $s_2 * s_3 < s_3 < s_1$ and $L2 = x_2 * x_3 = R4$.

L3R1 $s_1 = s_2$ and $s_2 < s_3$ yields $s_1 < s_3$ hence $L3 = x_1x_2 = R1$.

L3R2 $s_1 = s_2$ and $s_3 < s_2$ yields $s_3 < s_1$ hence $L3 = x_3 = R2$.

L3R3 $s_1 = s_2$ and $s_2 = s_3$ hence $L3 = (x_1x_2)x_3 = x_1(x_2x_3) = R3$.

L3R4 $s_1 = s_2$ and $s_2 \parallel s_3$ yields also $s_1 \parallel s_3$ hence $L3 = x_{s_1 * s_3} = x_{s_2 * s_3} = R4$, since $s_2 * s_3 < s_2 = s_1$.

L4R1 $s_1 \parallel s_2$ and $s_2 < s_3$ yields $s_1 < s_3$ since $\langle c \rangle$ is a CDC. Hence $L4 = x_{s_1 * s_2} = R1$ because $s_1 * s_2 < s_2 < s_3$.

L4R2 Here we get $s_3 < s_1$ and consequently $s_3 \leq s_1 * s_2$. Hence $L4 = x_3 = R2$.

L4R3 We have $s_1 \parallel s_2$ and $s_2 = s_3$ therefore $s_1 \parallel s_3$ and $L4 = x_{s_1 * s_2} = R3$ since $s_1 * s_2 < s_2 = s_3$.

L4R4 $s_1 \parallel s_2$ and $s_2 \parallel s_3$ yields $s_1 \parallel s_3$ since $\langle c \rangle$ is a CDC hence $s_1 * s_2 = s_2 * s_3 < s_1, s_3$. Consequently $L4 = x_{s_1 * s_2} = x_{s_2 * s_3} = R4$.

This proves that $(W, \mathcal{O})$ is a band. It is clear that $W/J \equiv \langle c \rangle$ and $(a, x) < (b, y)$ if and only if $a < b$, hence $(W, \mathcal{O})$ is a weak ordered band according to theorem 4.1.

**Theorem 4.3.** Let $B$ be a weak ordered band, then there are a CDC $(\langle c \rangle, *)$, rectangular bands $RB_a$, $a \in \langle c \rangle$, such that

$$B \equiv (W, \mathcal{O})$$

where $(W, \mathcal{O})$ is defined as shown above.
Proof. Since \( B \) is weak ordered we know that \( B/J \) is a CDC and \( B \) is a CDC of rectangular bands.

Now if \( e \in RB_a \) and \( f \in RB_b \) then \( ef \in RB_{a*b} \), where \( \ast \) denotes the multiplication in \( B/J \). More precisely if \( a < b \) then \( ef = e = fe \), since \( e < f \) if \( a < b \). Now suppose that \( a \parallel b \). Then we know that \( ef < e \) and \( ef < f \) since \( a*b < a \) and \( a \ast b < b \). Consequently \( ef = efe = fe \).

Moreover let \( e' \in RB_a \) and \( f' \in RB_b \) then
\[
e'f' = e'ee'f' = e' \underbrace{ef' e'}_{eRB_{a*b}} = ef' = ef'ff' = ef.
\]

But we can even show more. Let \( p \in RB_{a*b} \) then
\[
(ef)p = p \quad \text{since} \quad p < e \quad \text{and} \quad p < f
\]
we also have \( p(ef) = p \) and therefore
\[
 cf = (ef)p(ef) = p
\]
and \( RB_{a*b} \) consists only of one element.

These observations showed that \( B \cong (W, \leq) \) where
\[
W = \{(a,x) : a \in B/J, x \in RB_a\}
\]

Hence we established the required statement.

Example 4.4. Let \( c = (1, 2, 2) \) be a CDC with
\[
\langle c \rangle = \{(1, 0, 0), (1, 1, 0), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}.
\]

This CDC is given by the following diagram

![Diagram](image)
We choose isomorph rectangular bands $RB_a$ except for $a = (1, 0, 0)$ what must be a oneelementic set. $RB_a = \{x, y\}$ with $xy = yy = y$ and $yx = xx = x$ then we get

$$W = \{(1, 0, 0), 1),
((1, 1, 0), (1, 0, 0), y), ((1, 2, 0), x), ((1, 2, 0), y),
((1, 0, 1), x), ((1, 2, 1), y),
((1, 2, 2), x), ((1, 2, 2), y)\} =
\{w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$$

The natural partial order i given above. The multiplication table is:

<table>
<thead>
<tr>
<th></th>
<th>$w_0$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
<th>$w_5$</th>
<th>$w_6$</th>
<th>$w_7$</th>
<th>$w_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_0$</td>
<td>$w_0$</td>
<td>$w_0$</td>
<td>$w_0$</td>
<td>$w_0$</td>
<td>$w_0$</td>
<td>$w_0$</td>
<td>$w_0$</td>
<td>$w_0$</td>
<td>$w_0$</td>
</tr>
<tr>
<td>$w_1$</td>
<td>$w_0$</td>
<td>$w_1$</td>
<td>$w_2$</td>
<td>$w_0$</td>
<td>$w_1$</td>
<td>$w_1$</td>
<td>$w_1$</td>
<td>$w_1$</td>
<td>$w_1$</td>
</tr>
<tr>
<td>$w_2$</td>
<td>$w_0$</td>
<td>$w_1$</td>
<td>$w_2$</td>
<td>$w_0$</td>
<td>$w_2$</td>
<td>$w_2$</td>
<td>$w_2$</td>
<td>$w_2$</td>
<td>$w_2$</td>
</tr>
<tr>
<td>$w_3$</td>
<td>$w_0$</td>
<td>$w_1$</td>
<td>$w_2$</td>
<td>$w_3$</td>
<td>$w_3$</td>
<td>$w_3$</td>
<td>$w_3$</td>
<td>$w_3$</td>
<td>$w_3$</td>
</tr>
<tr>
<td>$w_4$</td>
<td>$w_0$</td>
<td>$w_1$</td>
<td>$w_2$</td>
<td>$w_3$</td>
<td>$w_4$</td>
<td>$w_4$</td>
<td>$w_4$</td>
<td>$w_4$</td>
<td>$w_4$</td>
</tr>
<tr>
<td>$w_5$</td>
<td>$w_0$</td>
<td>$w_1$</td>
<td>$w_2$</td>
<td>$w_3$</td>
<td>$w_4$</td>
<td>$w_5$</td>
<td>$w_6$</td>
<td>$w_6$</td>
<td>$w_6$</td>
</tr>
<tr>
<td>$w_6$</td>
<td>$w_0$</td>
<td>$w_1$</td>
<td>$w_2$</td>
<td>$w_3$</td>
<td>$w_4$</td>
<td>$w_5$</td>
<td>$w_6$</td>
<td>$w_7$</td>
<td>$w_8$</td>
</tr>
<tr>
<td>$w_7$</td>
<td>$w_0$</td>
<td>$w_1$</td>
<td>$w_2$</td>
<td>$w_3$</td>
<td>$w_4$</td>
<td>$w_5$</td>
<td>$w_6$</td>
<td>$w_7$</td>
<td>$w_8$</td>
</tr>
<tr>
<td>$w_8$</td>
<td>$w_0$</td>
<td>$w_1$</td>
<td>$w_2$</td>
<td>$w_3$</td>
<td>$w_4$</td>
<td>$w_5$</td>
<td>$w_6$</td>
<td>$w_7$</td>
<td>$w_8$</td>
</tr>
</tbody>
</table>

REFERENCES