QUARTER-SYMMETRIC METRIC CONNECTION IN AN SP-SASAKIAN MANIFOLD

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ABSTRACT

The object of the present paper is to study some properties of curvature tensor of a quarter-symmetric metric connection in an SP-Sasakian manifold.

1. INTRODUCTION

Let $M^n$ be an $n$-dimensional $C^n$ - manifold. If there exists a tensor field $F$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ in $M^n$ satisfying

$$\overline{X} = X - \eta(X) \xi, \quad \overline{X} = F(X), \quad \eta(\xi) = 1,$$  \hspace{1cm} (1.1)

then $M^n$ is called an almost paracontact manifold.

Let $g$ be the Riemannian metric satisfying

$$\eta(X) = g(X, \xi)$$  \hspace{1cm} (1.2)

$$\eta(FX) = 0, \quad f_\xi = 0, \quad \text{rank } (F) = n-1$$  \hspace{1cm} (1.3)

$$g(FX, FY) = g(X, Y) - \eta(X) \eta(Y)$$  \hspace{1cm} (1.4)

The set $(F, \xi, \eta, g)$ satisfying (1.1), (1.2), (1.3) and (1.4) is called an almost paracontact Riemannian structure. The manifold with such a structure is called an almost paracontact Riemannian manifold [3].

If we define $\overline{F}(X,Y) = -g(X,Y)$, then in addition to the above relations the followings are satisfied;

$$\overline{F}(X,Y) = F(Y,X)$$  \hspace{1cm} (1.5)

$$\overline{F}(\overline{X}, \overline{Y}) = F(X,Y)$$  \hspace{1cm} (1.6)
Let us consider an $n$-dimensional differentiable manifold $M$ with a positive definite metric $g$ which admits a $1$-form $\eta$ satisfying

$$ (\nabla_X \eta) (Y) - (\nabla_Y \eta) (X) = 0 $$

(1.7)

and

$$ (\nabla_X \nabla_Y \eta) (Z) = - g(X,Z) \eta(Y) - g(X,Y) \eta(Z) + 2\eta(X) \eta(Y) \eta(Z) $$

(1.8)

where $\nabla$ denotes the covariant differentiation with respect to $g$. Moreover, if we put

$$ \eta(X) = g(X,\xi), \quad \nabla_X \xi = \bar{X}, $$

(1.9)

then it can be easily verified that the manifold in consideration becomes an almost contact Riemannian manifold. Such a manifold is called a P-Sasakian manifold [1].

For a P-Sasakian manifold $M$, the following relations hold;

$$ \eta(\mathcal{R}(X,Y)Z) = g(X,Z) \eta(Y) - g(Y,Z) \eta(X), $$

(1.10)

$$ S(X,\xi) = -(n-1) \eta(X), $$

(1.11)

where $\mathcal{R}$ and $S$ are the curvature tensor and the Ricci tensor respectively.

Now, we consider an $n$-dimensional differentiable manifold $M$ with a Riemannian metric $g$ which admits a $1$-form $\eta$ satisfying

$$ (\nabla_X \eta) (Y) = - g(X,Y) + \eta(X) \eta(Y). $$

(1.12)

By putting $\eta(X) = g(X,\xi)$ and $(\nabla_X \eta)(Y) = \mathcal{F}(X,Y)$, we can easily show that the manifold in consideration is a P-Sasakian manifold. Such a manifold is called an SP-Sasakian manifold [1]. Thus for such a SP-Sasakian manifold, we have

$$ \mathcal{F}(X,Y) = - g(X,Y) + \eta(X) \eta(Y) $$

(1.13)

A linear connection $\widetilde{\nabla}$ in a Riemannian manifold $M^n$ is said to be a quarter-symmetric connection if its torsion tensor $T$ satisfies
\[ T(X,Y) = \eta(Y) \, \varphi(X) - \eta(X) \, \varphi(Y) \]  

(1.14)

where \( \eta \) is a 1-form and \( \varphi \) is a (1,1) tensor field [2]. A linear connection \( \nabla \) is called a metric connection, iff

\[ \left( \nabla_{\mathbf{g}} \right)(YZ) = 0 \]  

(1.15)

A linear connection \( \nabla \) satisfying (1.14) and (1.15) is called a quarter-symmetric metric connection [5].

If \( \varphi(X) = X \), then the connection is called a semi-symmetric metric connection [5]. The semi-symmetric metric connection in an SP-Sasakian manifold have been studied by Sinha and Kalpana [4]. In the present paper we have studied with the quarter-symmetric metric connection in an SP-Sasakian manifold. In section 2 we have deduced the expressions for the curvature tensor and the Ricci tensor of \( M^n \) with respect to the quarter-symmetric metric connection. Some properties of the curvature tensor with respect to the quarter-symmetric metric connection have been studied. In general, the Ricci tensor of the quarter-symmetric metric connection is not symmetric. Here it is proved that in an SP-Sasakian manifold the Ricci tensor of the quarter-symmetric metric connection is symmetric. Also, in general, the conformal curvature tensors of the quarter-symmetric metric connection and the Riemannian connection are not equal. Finally it is proved that in an SP-Sasakian manifold the conformal curvature tensors of the quarter-symmetric metric connection and the Riemannian connection are equal and also it is proved that if the curvature tensor of the quarter-symmetric metric connection vanishes then the manifold is conformally flat.

2. CURVATURE TENSOR

We consider \( \varphi(X) \) as a contact structure \( F(X) = \bar{X} \) in equation (1.14). The manifold \( M^n \) is considered to be an SP-Sasakian manifold. The equation (1.14) and (1.15) can be written as

\[ T(X,Y) = \eta(Y)\bar{X} - \eta(X)\bar{Y} \]  

(2.1)

\[ \left( \nabla_{\mathbf{g}} \right)(YZ) = 0 \]  

(2.2)
Let $\widetilde{\nabla}$ be a linear connection and $\nabla$ be a Riemannian connection such that

$$\widetilde{\nabla}_X Y = \nabla_X Y + U(X,Y)$$  \hspace{1cm} (2.3)

where $U$ is a tensor of type $(1,2)$. For $\widetilde{\nabla}$ to be a quarter-symmetric metric connection in $M^n$, we have from

$$U(X,Y) = \frac{1}{2} \left[ T(X,Y) + T'(X,Y) + T'(Y,X) \right]$$  \hspace{1cm} (2.4)

where

$$g(T'(X,Y),Z) = g(T(Z,X),Y)$$  \hspace{1cm} (2.5)

(See [5]).

From (2.1) and (2.5) we get

$$T'(X,Y) = \eta(X)\overline{Y} - F(X,Y)\xi$$  \hspace{1cm} (2.6)

where $\xi = \eta(\overline{X}), \eta$ is a 1-form and $\xi$ is the associated vector field.

From (2.1), (2.4) and (2.6) we have

$$U(X,Y) = \eta(Y)\overline{X} - F(X,Y)\xi$$  \hspace{1cm} (2.7)

From (2.3) and (2.7) we get

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\overline{X} - F(X,Y)\xi$$  \hspace{1cm} (2.8)

Hence a quarter-symmetric metric connection $\widetilde{\nabla}$ in an SP-Sasakian manifold is given by (2.8).

Let $\widetilde{R}$ and $R$ be the curvature tensors of the connections $\widetilde{\nabla}$ and $\nabla$ respectively. Then we have

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - [\widetilde{\nabla}_X, \widetilde{\nabla}_Y]Z$$  \hspace{1cm} (2.9)

and

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - [\nabla_X, \nabla_Y]Z$$  \hspace{1cm} (2.10)

Using (2.8) and (2.10) in (2.9) we have
\[ \tilde{\mathcal{R}}(X,Y)Z = R(X,Y)Z + 3F(X,Z)\bar{Y} - 3F(Y,Z)\bar{X} \]
\[ + \left[ \left( \nabla_X F \right)(Y) - \left( \nabla_Y F \right)(X) \right] \eta(Z) - \]
\[ + \left[ \left( \nabla_X F \right)(YZ) - \left( \nabla_Y F \right)(XZ) \right] \xi \]  
(2.11)

Equation (2.11) can be written as
\[ \tilde{\mathcal{R}}(X,Y,Z,U) = \mathcal{R}(X,Y,Z,U) + 3F(X,Z)F(Y,U) - 3F(Y,Z)F(X,U) \]
\[ + \left[ \left( \nabla_X F \right)(Y,U) - \left( \nabla_Y F \right)(X,U) \right] \eta(Z) - \]
\[ - \left[ \left( \nabla_X F \right)(YZ) - \left( \nabla_Y F \right)(XZ) \right] \eta(U) \]  
(2.12)

where \( \tilde{\mathcal{R}}(X,Y,Z,U) = g(\tilde{\mathcal{R}}(X,Y)Z,U) \) and \( \mathcal{R}(X,Z,Y,U) = g(\mathcal{R}(X,Y)Z,U) \)

Since
\[ (\nabla_Y F)(Y,Z) = \nabla_X F(Y,Z) - F(\nabla_Y Y, Z) - F(Y, \nabla_X Z) \]  
(2.13)

then comparing with (2.13) we get
\[ (\nabla_Y F)(Y,Z) = F(X,Y) \eta(Z) + F(X,Z) \eta(Y) \]

So
\[ (\nabla_Y F)(Y,Z) - (\nabla_X F)(X,Z) = F(X,Z) \eta(Y) - F(Y,Z) \eta(X) \]  
(2.14)

From (1.13), (2.14) and (2.12) we get
\[ \tilde{\mathcal{R}}(X,Y,Z,U) = \mathcal{R}(X,Y,Z,U) + 3g(X,Z)g(Y,U) - 3g((Y,Z)g(X,U) \]
\[ - 2g(X,Y) \eta(Y) \eta(U) - 2g(Y,U) \eta(X) \eta(Z) \]
\[ + 2g(Y,Z) \eta(X) \eta(U) + 2g(X,U) \eta(Y) \eta(Z) \]  
(2.15)

A relation between the curvature tensor of \( M^n \) with respect to the quarter-symmetric metric connection \( \tilde{\nabla} \) and the Riemannian connection \( \nabla \) is given by the equation (2.15). Putting \( X = U = e_i \) in (2.15) where \( \{e_i\} \) is an orthonormal basis of the tangent space at any point of the manifold and taking summation over \( i, 1 \leq i \leq n \) we get
\[ \tilde{\mathcal{S}}(Y,Z) = S(Y,Z) - (3n-5)g(Y,Z) + 2(n-2)\eta(Y)\eta(Z) \]  
(2.16)

where \( \tilde{\mathcal{S}} \) and \( S \) are the Ricci tensors of the connection \( \tilde{\nabla} \) and \( \nabla \)
respectively. Again, putting $Y = Z = e_i$ in (2.16) we get

$$
\tilde{r} = r - (n-1)(3n-4)
$$

(2.17)

where $\tilde{r}$ and $r$ are the scalar curvatures of the connections $\tilde{\nabla}$ and $\nabla$ respectively.

**Theorem 1.** For a SP-Sasakian manifold $M$ with quarter-symmetric metric connection $\tilde{\nabla}$, we have

(a) $\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0$
(b) $\tilde{R}(X,Y,Z,U) + \tilde{R}(X,Y,U,Z) = 0$
(c) $\tilde{R}(X,Y,Z,U) - \tilde{R}(Z,U,X,Y) = 0$
(d) $\tilde{R}(X,Y,Z,\xi) = 2\tilde{R}(XY,Z,\xi)$
(e) $\tilde{S}(X,\xi) = 2S(X,\xi)$

**Proof.** Using (2.15) and the first Binachi identity with respect to the Riemannian connection, we have (a). From (2.15) we get (b) and (c). Putting $U = \xi$ in (2.15) and using (1.10) we get (d). Putting $Y = Z = e_i$ in (d) and taking summation over $i$, we get (e).

**Theorem 2.** In an SP-Sasakian manifold $M$ the Ricci tensor of the quarter-symmetric metric connection is symmetric.

**Proof.** The proof of the theorem obviously follows from (2.16).

Weyl conformal curvature tensor $C$ of type $(0,4)$ of $M^n$ with respect to the Riemannian connection is given by

$$
C(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{n-2} \left[ S(Y,Z) g(X,U) - S(X,Z) g(Y,U) + S(X,Y) g(Z,U) - S(Z,Y) g(X,U) \right] + \frac{r}{(n-1)(n-2)} \left[ g(Y,Z) g(X,U) - g(X,Z) g(Y,U) \right]
$$

(2.18)

Analogous to this definition, we define conformal curvature tensor of $M^n$ with respect to the quarter-symmetric metric connection by

$$
\tilde{C}(X,Y,Z,U) = \tilde{R}(X,Y,Z,U) - \frac{1}{n-2} \left[ \tilde{S}(Y,Z) g(X,U) - \tilde{S}(X,Z) g(Y,U) \right]
$$
\begin{align*}
+ \tilde{S}(X,U) \ g(Y,Z) - \tilde{S}(Y,U) \ g(X,Z) + \\
+ \frac{\tilde{r}}{(n-1) (n-2)} \ [g(Y,Z) \ g(X,U) - G(X,Z) \ g(Y,U)]
\end{align*}
(2.19)

From (2.15), (2.16), (2.17), (2.18) and (2.19), we have

\[ \tilde{\mathcal{C}}(X,Y,Z,U) = \mathcal{C}(X,Y,Z,U) \]  
(2.20)

Hence we can state the following theorem.

**Theorem 3.** In an SP-Sasakian manifold the conformal curvature tensors of the quarter-symmetric metric connection and the Riemannian connection are equal.

Let us now consider \( \tilde{R} = 0 \). Then we have \( \tilde{S} = 0 \) and \( \tilde{r} = 0 \) and hence from (2.19) we get \( \tilde{C} = 0 \). So, from (2.20) we get \( C = 0 \).

Thus we have the following theorem:

**Theorem 4.** If in an SP-Sasakian manifold the curvature tensor of a quarter-symmetric metric connection vanishes, then the manifold is conformally flat.

If \( \tilde{S} = 0 \), then from (2.16) we get

\[ S(Y,Z) = (3n-5) \ g(Y,Z) - 2(n-2) \ \eta(Y) \ \eta(Z) \]  
(2.21)

Since \( \tilde{S} = 0 \), \( \tilde{r} = 0 \), and so from (2.17) we get

\[ r = (n-1) (3n-4) \]  
(2.22)

From (2.15), (2.18), (2.21) and (2.22) we get

\[ \tilde{\mathcal{R}}(X,Y,Z,U) = \mathcal{C}(X,Y,Z,U) \]

Hence we can state the following theorem:

**Theorem 5.** If in an SP-Sasakian manifold the Ricci tensor of a quarter-symmetric metric connection \( \tilde{\nabla} \) vanishes, then the curvature tensor of \( \tilde{\nabla} \) is equal to the conformal curvature tensor of the manifold.

From Theorem 4 and Theorem 5 we have the following theorem:
Theorem 6. If in an SP-Sasakian manifold the Ricci tensor of a quarter-symmetric metric connection $\tilde{\nabla}$ vanishes, then the manifold is conformally flat iff the curvature tensor with respect to the quarter-symmetric metric connection vanishes.

REFERENCES


