PROPERTIES OF 2-DIMENSIONAL SPACE-LIKE RULED SURFACES IN THE MINKOWSKI SPACE $\mathbb{R}_1^n$

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ABSTRACT

In this paper we find new characteristic properties for 2-dimensional ruled surface $M$ in $\mathbb{R}_1^n$ and give the sufficient and necessary conditions for which the space-like ruled surface $M$ is to be total geodesic. In addition, some characterisation which is the well-known for the ruled surfaces in the Euclidean 3-space was generalized for the space-like ruled surfaces in $\mathbb{R}_1^n$.

1. INTRODUCTION

We shall assume throughout this paper that all manifolds, maps, vector fields, etc... are differentiable of class $C^\infty$. Consider a general submanifold $M$ of the Minkowski space $\mathbb{R}_1^n$. Suppose that, $\overrightarrow{D}$ is the Levi-Civita connection of Minkowski space $\mathbb{R}_1^n$, while $D$ is the Levi-Civita connection of Semi Riemann manifold $M$. If $X$ and $Y$ are the vector fields of $M$ and if $V$ is second fundamental form of $M$, we have by decomposing $D_X Y$ in a tangential and normal component.

$$\overrightarrow{D}_X Y = D_X Y + V(X,Y) \quad (1.1)$$

The equation (1.1) is called Gauss equation, [1].

If $\xi$ is any normal vector filed on $M$, we find the Weingarten equation by decomposing $\overrightarrow{D}_X \xi$ in a tangential and normal component

$$\overrightarrow{D}_X \xi = -A_\xi + D_X \xi. \quad (1.2)$$
Aₐ determines at each point a self-adjoint linear map and Dₐ is a metric connection in the normal bundle χₐ(M). We use the same notation Aₐ for the linear map and the matrix of the linear map, [1].

A normal vector field ξ is called parallel in the normal bundle χₐ(M) if we have Dₓξ = 0 for each vector X. If η is a normal unit vector at the point p ∈ M, then

\[ G(p, η) = \det A_η \]  \hspace{1cm} (1.3)

is the Lipschitz-Killing curvature of M at p in direction η, [2].

Suppose that X and Y are vector fields on M, while ξ is a normal vector field on χₐ(M). If the standard metric tensor of \( \mathbb{R}^n \) is denoted by \(<\cdot, \cdot>\) then we have

\[ \langle Dₓ Y, ξ \rangle = \langle V(X, Y) ξ \rangle \]  \hspace{1cm} (1.4)

and

\[ \langle Dₓ X, ξ \rangle = \langle A_ξ(X), Y \rangle . \]  \hspace{1cm} (1.5)

From the above equations we obtain

\[ \langle V(X, Y) ξ \rangle = \langle A_ξ(X), Y \rangle \]  \hspace{1cm} (1.6)

If ξ₁, ξ₂, ..., ξₙ₂ constitute an orthonormal base field of the normal bundle χₐ(M), then we set

\[ \langle V(X, Y) ξ_j \rangle = V_j(X, Y) \]  \hspace{1cm} (1.6)

or

\[ V(X, Y) = \sum_{j=1}^{n²} V_j(X, Y) ξ_j. \]  \hspace{1cm} (1.7)

The mean curvature vector H of M at the point p is given by

\[ H = \sum_{j=1}^{n²} \frac{\text{tr} A_ξ}{2} ξ_j. \]  \hspace{1cm} (1.8)

||H|| is the mean curvature. If H = 0 at each point p of M, then M is said to be minimal, [1].
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2. 2-DIMENSIONAL SPACE-LIKE RULED SURFACES IN $\mathbb{R}^n_1$

Let $\alpha$ be a space-like curve and $e(s)$ be a space-like unit vector on the generators in $\mathbb{R}^n_1$. If the space-like base curve $\alpha$ is an orthogonal trajectory of the generators then we get a 2-dimensional ruled surface $M$. This ruled surface is called 2-dimensional space-like ruled surface and represented by

$$\psi(s,v) = a(s) + v e(s).$$

**Definition 2.1:** Let $M$ be 2-dimensional space-like ruled surface in $\mathbb{R}^n_1$ and $V$ be second fundamental form of $M$. If $V(X,X) = 0$ for all $X \in \chi(M)$ then $X$ is called an asymptotic vector field on $M$.

**Theorem 2.1:** Let $M$ be 2-dimensional space-like ruled surface in $\mathbb{R}^n_1$. Then the generators of $M$ are asymptotics and geodesics of $M$.

**Proof:** Since the generators are the geodesics of $\mathbb{R}^n_1$, we have

$$\overline{D}e = 0.$$  

If we set this in the Gauss equation, we get

$$D_e e + V(e,e) = 0$$  

or

$$D_e e = -V(e,e).$$

Since $D_e e \in \chi(M)$ and $V(e,e) \in \chi^1(M)$ we get $D_e e = 0$ and $V(e,e) = 0$.

Therefore the generators of $M$ are the asymptotics and geodesics of $M$.

Suppose that $\{e_1, e_i\}$ is an orthonormal base field of the tangential bundle $\chi(M)$ and $\{\xi_1, \xi_2, ..., \xi_{n-2}\}$ is an orthonormal bundle $\chi^1(M)$. Then we have the following equations.

$$\overline{D}_{\xi} \xi_j = a_{1j}^1 e + a_{1j}^2 e_1 + \sum_{i=1}^{n-2} b_{1i}^j \xi_i, \quad 1 \leq j \leq n-2$$

$$\overline{D}_{\xi_1} \xi_j = a_{2j}^1 e + a_{2j}^2 e_1 + \sum_{i=1}^{n-2} b_{2i}^j \xi_i, \quad 1 \leq j \leq n-2$$  \hspace{1cm} (2.1)

From these equations we observe that

$$a_{21}^j = -a_{12}^j, \quad a_{11}^j = 0, \quad 1 \leq j \leq n$$

and
\[
A_{\xi_j} = \begin{bmatrix}
0 & a_{12}^j \\
-a_{12}^j & 0
\end{bmatrix}.
\] (2.2)

Then we have the following corollary.

**Corollary 2.1:** The matrix \(A_{\xi_j}\) is corresponding to the shape operator of \(M\) and \(A_{\xi_j}\) is a symmetric matrix in the sense of Lorentz.

**Corollary 2.2:** The Lipschitz-Killing curvature at \(p \in M\) in the direction of \(\xi_j\) is given by

\[
G(p,\xi_j) = -(a_{12})^j.
\]

From (2.1) we have

\[
a_{12}^j = \langle \overline{D}_{\varepsilon_1} \xi_j, \varepsilon_1 \rangle = -\langle \xi_j, \overline{D}_{\varepsilon_1} \varepsilon_1 \rangle
\] (2.3)

and

\[
\langle \overline{D}_{\varepsilon_1} \varepsilon_1 \rangle = -\langle \varepsilon_1, \overline{D}_{\varepsilon_1} \varepsilon_1 \rangle = 0
\] (2.4)

while

\[
\langle \overline{D}_{\varepsilon_1} \varepsilon_1 \rangle = -\langle \varepsilon_1, \overline{D}_{\varepsilon_1} \varepsilon_1 \rangle = 0.
\] (2.5)

From (2.4) and (2.5) we observe that

\[
\overline{D}_{\varepsilon_1} \varepsilon_1 \in \chi^1(M) \text{ or } \overline{D}_{\varepsilon_1} = V(\varepsilon_1).\]

Because of (2.3) we have

\[
\overline{D}_{\varepsilon_1} = V(\varepsilon_1) = \sum_{j=1}^{n^2} \varepsilon_j \langle \xi_j, \overline{D}_{\varepsilon_1} \varepsilon_1 \rangle \xi_j = -\sum_{j=1}^{n^2} \varepsilon_j a_{12}^j \xi_j
\] (2.6)

\[
\varepsilon_j = \langle \xi_j, \xi_j \rangle = \begin{cases}
-1, & \xi_j \text{ time-like} \\
1, & \xi_j \text{ space-like}
\end{cases}.
\]

Because of (1.4) and (2.1) we find

\[
a_{22}^j = \langle \overline{D}_{\xi_j} \varepsilon_1, \varepsilon_1 \rangle = -\langle A_{\xi_j} (\varepsilon_1), \varepsilon_1 \rangle = -\langle \mathcal{V}(\varepsilon_1, \varepsilon_1), \xi_j \rangle
\] (2.7)

and

\[
\text{tr } A_{\xi_j} = -a_{22}^j = \langle \mathcal{V}(\varepsilon_1, \varepsilon_1), \xi_j \rangle, \quad 1 \leq j \leq n^2.
\] (2.8)
Theorem 2.2: Let $M$ be 2-dimensional space-like ruled surface in $\mathbb{R}^n_1$ and $\{e_1,e\}$ be the orthonormal base field of the tangential bundle $\chi(M)$. Then the Gauss curvature $G$ can be given as follows

$$G = \langle \overline{D}_e e_1, \overline{D}_e e_1 \rangle.$$

Proof: Let $R$ be the Riemannian curvature tensor field of $M$. In this case we get

$$G = \langle R(e_1,e) e, e_1 \rangle, \quad [3]. \quad (2.9)$$

By combining (2.9) and $V(e,e) = 0$ we are faced with

$$G = \langle V(e,e_1), V(e,e_1) \rangle$$

or

$$G = \langle \overline{D}_e e_1, \overline{D}_e e_1 \rangle.$$

From the above Theorem 2.2 Corollary 2.2 and the equation (2.6) we have the following corollaries.

Corollary 2.3: The Gauss curvature of $M$ with respect to the elements of $A_{i,j}$.

$$G = \sum_{j=1}^{n^2} \varepsilon_j \left( \frac{\langle e_1 \rangle}{2} \right)^2. \quad (2.11)$$

Corollary 2.4: A space-like ruled surface $M$ is developable if and only if the Lipschitz-Killing curvature is zero at each point.

Theorem 2.3: Let $M$ be a 2-dimensional space-like ruled surface in $\mathbb{R}^n_1$. The mean curvature of $M$ is

$$H = \frac{1}{2} \varepsilon_j V(e_1,e_1).$$

Proof: From (1.8) we know that

$$H = \sum_{j=1}^{n^2} \text{tr} \frac{A_{i,j}}{2} \xi_j. \quad (2.12)$$

For the matrix $A_{i,j}$ given (2.2) we find

$$\text{tr} A_{i,j} = -a_{i,2}^j.$$
If we substitute (2.8) in (1.8) we get
\[ H = \frac{1}{2} \varepsilon_j V(e_1, e_j). \]

**Theorem 2.4:** Let \( M \) be 2-dimensional space-like ruled surface in \( \mathbb{R}^n_1 \). \( M \) is developable and minimal iff \( M \) is total geodesic.

**Proof:** We assume that \( M \) is developable and minimal. If \( X, Y \in \chi(M) \), we have \( X = ae + be_1 \) and \( Y = ce + de_1 \).

Therefore we get
\[ V(X, Y) = ac V(e, e) + (ad + bc) V(e, e_1) + bd V(e_1, e_1). \]

Because of Theorem 2.1 and minimality of \( M \) we have \( V(e, e) = 0 \) and \( V(e_1, e_1) = 0 \). Moreover, since \( M \) is developable \( \bar{D}_e e_1 = 0 \). Thus we can write \( V(e, e_1) = 0 \) and \( V(X, Y) = 0 \) for all \( X, Y \in \chi(M) \).

Now suppose that \( V(X, Y) = 0, \ \forall X, Y \in \chi(M) \). Then we have \( V(e, e) = 0 \), \( V(e, e_1) = 0 \). Because of Theorem 2.1 we have
\[ \langle \bar{D}_e e_1, e \rangle = 0 \text{ and } \langle \bar{D}_e e_1, e_1 \rangle = 0. \]

This means that \( \bar{D}_e e_1 \) is a normal vector field or \( \bar{D}_e e_1 = V(e, e_1) \).

Therefore we have \( \bar{D}_e e_1 = 0 \). This implies that \( M \) is developable and \( V(e, e_1) = 0 \) implies that \( M \) is minimal.

Let \( M \) be 2-dimensional space-like ruled surface in \( \mathbb{R}^n_1 \) and \( e \) be unit space-like vector field of the generator. Then we have the following equations of covariant derivative of the orthonormal base field \( \{ e, e_1, \xi_1, \xi_2, ..., \xi_{n-2} \} \).

\[
\begin{align*}
\bar{D}_{e_1} e_1 &= c_{11} e_1 + c_{12} e + c_{13} \xi_1 + ... + c_{1n} \xi_{n-2} \\
\bar{D}_{e_1} e &= c_{21} e_1 + c_{22} e + c_{23} \xi_1 + ... + c_{2n} \xi_{n-2} \\
\bar{D}_{e_1} \xi_1 &= c_{31} e_1 + c_{32} e + c_{33} \xi_1 + ... + c_{3n} \xi_{n-2} \\
&... \\
\bar{D}_{e_1} \xi_{n-2} &= c_{n1} e_1 + c_{n2} e + c_{n3} \xi_1 + ... + c_{nn} \xi_{n-2}.
\end{align*}
\]
If we write these equations in the matrix form we get

\[
\begin{bmatrix}
\overline{D}_{\xi} e_1 \\
\overline{D}_{\xi} e \\
\overline{D}_{\xi} \xi \\
\vdots \\
\overline{D}_{\xi} \xi_{n-2}
\end{bmatrix}
= 
\begin{bmatrix}
0 & c_{12} & c_{13} & \cdots & c_{1n} \\
- c_{12} & 0 & c_{23} & \cdots & c_{2n} \\
- \varepsilon_1 c_{13} & - \varepsilon_1 c_{23} & 0 & \cdots & c_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
- \varepsilon_1 c_{1n} & - \varepsilon_1 c_{2n} & - c_{3n} & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e \\
\xi_1 \\
\vdots \\
\xi_{n-2}
\end{bmatrix}
\]

(2.13)

Theorem 2.5: Let \( M \) be a 2-dimensional space-like ruled surface in \( \mathbb{R}^n \), \( \{ e_1, e \} \) be an orthonormal base field of the tangential bundle \( \chi(M) \) and \( \alpha(s) \) be an orthonormal trajectory of the generators of \( M \). Then the following propositions are equivalent.

i) \( M \) is developable

ii) The Lipschitz-Killing curvature

\( G(p, \xi_j) = 0 \), \( 1 \leq j \leq n-2 \)

iii) The Gauss curvature \( G = 0 \).

iv) In the equation (2.13), \( e_{2k} = 0 \), \( 3 \leq k \leq n \).

v) \( A_{\xi_j}(e) = 0 \)

vi) \( \overline{D}_{\xi_j} e \in \chi(M) \).

Proof: i \( \Rightarrow \) ii : We assume that \( M \) is developable, since \( a_{11}^j = 0 \) in (2.1), \( 1 \leq j \leq n-2 \), the Lipschitz-Killing curvature at point \( p \) in the direction of \( \xi_j \) is given by

\[
G(p, \xi_j) = - \left( a_{12}^j(p) \right)^2 = 0 \quad , \quad 1 \leq j \leq n-2 .
\]

Because of (2.6) and since \( M \) is developable we have

\[
\overline{D}_{\xi_j} e_1 = - \sum_{j=1}^{n-2} \varepsilon_j (a_{12}^j) \xi_j = 0 .
\]

So we find \( G(p, \xi_j) = 0 \), \( 1 \leq j \leq n-2 \).

ii \( \Rightarrow \) iii : Let \( G(p, \xi_j) = 0 \), \( 1 \leq j \leq n-2 \).
Since we have

\[ G(p) = -\sum_{j=1}^{n^2} \xi_j \xi_j^* \quad \forall \ p \in M \]

we observe that \( G = 0, \forall \ p \in M. \)

\( \text{iii} \Rightarrow \text{iv} \) : Suppose that \( G = 0, \forall \ p \in M. \) Then because of (2.11) we have \( a^j_{12} = 0, \ 1 \leq j \leq n-2. \) So \( \overline{D}_e \xi_j \) has no component in the direction \( e. \) Hence we observe that \( c_{2k} = 0, \ 3 \leq k \leq n, \) in the equation (2.13).

\( \text{iv} \Rightarrow \text{v} \) : Suppose that \( c_{2k} = 0, \ 3 \leq k \leq n, \) in the equation (2.13). That shows that \( \overline{D}_e \xi_j \) has no component in the direction \( e. \) Thus we have in the equation (2.1), \( a^j_{12} = 0, \ 1 \leq j \leq n-2. \)

Moreover, since \( a^j_{11} = \overline{D}_e \xi_j e \) and \( \langle \xi_j, D_e \rangle = 0 \) and because of the Weingarten equation we find

\[ A_{e_j}(e) = 0, \ 1 \leq j \leq n-2. \]

\( \text{v} \Rightarrow \text{vi} \) : Let \( A_{e_j}(e) = 0. \) Then, from the Weingarten equation, we have \( a^j_{11} = 0, \ a^j_{12} = 0, \ 1 \leq j \leq n-2. \) Moreover, \( \langle e, e_j \rangle = 0 \) implies

\[ \overline{D}_e e_j e = -\langle e_j, D_e e \rangle \]  \hspace{1cm} (2.14)

If we se equations 2.1 and last equations we get

\[ \overline{D}_e e_j e = -\langle e, D e_j e \rangle = -a^j_{12} \]

and

\[ \overline{D}_e e_j e = 0. \]

From the last equation we have

\[ \overline{D}_e e \in \chi(M). \]

\( \text{vi} \Rightarrow \text{i} \) : Let \( \overline{D}_e e \in \chi(M). \) Then from the equation (2.14), we get

\[ \langle \overline{D}_e e, e_j \rangle = -a^j_{12} = 0, \ 1 \leq j \leq n-2. \]  On the other hand, \( e[\langle e_j, e \rangle] = e \) \[1\] implies that \( \overline{D}_e e_j e = 0 \) and \( e[\langle e_j, e \rangle] = e[0] \) implies that \( \overline{D}_e e_j e = 0 \) (Since the generators are the geodesics of \( R^n, \) we have \( \overline{D}_e e = 0). \) Thus \( \overline{D}_e e_j \in \chi(M). \)
Because of (2.6) and since $a_{12}^j = 0, \ 1 \leq j \leq n-2$, we write that $\overline{D} e_1 = 0$.

This means that the tangent planes of $M$ constant along the generator $e$ of $M$, i.e. $M$ is developable.

**Corollary 2.5:** Let $M$ be a 2-dimensional space-like ruled surface in $\mathbb{R}^n_1$ with a Gauss curvature being zero. If $M$ is minimal, then $c_{sk} = 0, \ 1 \leq s \leq 2, \ 3 \leq k \leq n$, in the (2.13).

**Proof:** Let $M$ be minimal. Then from the equation (2.12) we have $V(e_1,e_1) = 0$. If this result is set in the Gauss equation, we find

$$\overline{D} e_1 = D e_1.$$ 

This means that $\overline{D} e_1$ has no component in $\chi^1(M)$. Therefore we have

$$C_{ik} = 0, \ 3 \leq k \leq n. \quad (2.15)$$

in the equation (2.13). On the other hand, since $G = 0$, by hypothesis, and from the Theorem 2.5 we know that $C_{2k} = 0, \ 3 \leq k \leq n$. If we consider this together with (2.15) we observe that $C_{sk} = 0, \ 1 \leq s \leq 2, \ 3 \leq k \leq n$.

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