SOME CONVOLUTION ALGEBRAS AND THEIR MULTIPLIERS

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ABSTRACT

Let $G$ be a locally compact Abelian group (nondiscrete and non compact) with dual group $\hat{G}$. For $1 \leq p < \infty$, $A_p(G)$ denotes the vector space of all complex-valued functions in $L^p(G)$ whose Fourier transforms $\hat{f}$ belong to $L^p(\hat{G})$. Research on the spaces $A_p(G)$ was initiated by Warner [20] and Larsen, Liu and Wang [14]. Later several generalizations of these spaces to the weighted case was given by Gürkanlı [6], Feichtinger and Gürkanlı [4] and Fischer, Gürkanlı and Liu [5]. One of these generalization is the space $A_{p,\omega}(G)$, [4]. Also the multipliers of $A_p(G)$ were discussed in some papers such as [14], [1], [13], [3], [9] and proved that the space of multipliers of $A_p(G)$ is the space of all bounded complex-valued regular Borel measures on $G$.

In the present paper we discussed the multipliers of the Banach algebra $A_{p,\omega}(G)$ and proved that under certain conditions for given any multiplier $T$ of $A_{p,\omega}(G)$ there exists a unique pseudo measure $\sigma$ such that $Tf = \sigma \ast f$ for all $f \in A_{p,\omega}(G)$.

1. INTRODUCTION

Let $G$ be a locally compact Abelian group with dual group $\hat{G}$ and let $dx$ and $d\hat{x}$ be Haar measures on these groups respectively. We denote by $K(G)$ the vector spaces of continuous functions on $G$ with compact support and $K_c(G)$ the subclass of those functions in $K(G)$ whose supports are contained in $C$. For functions in $L^1(G)$ the Fourier Transform is denoted by $\hat{f}$ or $Ff$. It is known that $\hat{f}$ is continuous on $\hat{G}$ which, vanish at infinity and the inequality $\|\hat{f}\|_\infty \leq \|f\|_1$ is satisfied ([16], 1.2.4. Theorem). We will denote the space of pseudo-measures by $A'(G)$, ([11], pp.97).

We set for $1 \leq p < \infty$,

$$L^p_w(G) = \left\{ f \mid f, w \in L^p(G) \right\},$$
where \( w \) is the Beurling's weight function on \( G \), i.e. \( w \) is a continuous function satisfying \( w(x) \geq 1 \) and \( w(x+y) \leq w(x) \cdot w(y) \) for all \( x, y \in G \). It is known that \( L^p_w(G) \) is a Banach space under the norm
\[
\| f \|_{p,w} = \left[ \int_G |f(x)|^p \cdot w^p(x) \, dx \right]^{1/p}
\]
\( L^1_w(G) \) is called a Beurling algebra \([15]\). In some parts of the present paper it is used an extra condition on \( W \): A weight \( w \) is said to satisfy the Beurling-Domar condition (Shortly. (BD)) if one has
\[
\sum_{n \geq 1} n^{-2} \log (w(nx)) < \infty
\]
for all \( x \in G \), \([2]\).

It is known that regular maximal ideal space of \( L^1(G) \) can be identified with the space of all generalized characters \( \eta \) on \( G \) such that \( \eta \in L^\infty_{\omega}(G) \) and \( \eta \leq \omega(x1.a.e. \] \([19]\). If \( w \) satisfies the (B.D) condition the regular maximal ideal space of \( L^1(G) \) is equal to the dual group \( \hat{G} \). (c.g[2] pp.15 and Theorem 2.11).

Now we set
\[
\bigwedge^W_K(G) = \left\{ f \in L^1_w(G) \mid f \in K(\hat{G}) \right\},
\]
\[
\bigwedge^W_K(L(G)) = \left\{ f \in \bigwedge^W_K(G) \mid f \in K_L(\hat{G}) \right\}
\]
where \( \hat{L} \subset (\hat{G}) \). Again \( A^1(G) \) will denote the linear subspace of \( L^1(G) \) consisting of those \( f \in L^1(G) \) such that \( \hat{f} \in L^1(\hat{G}) \). It is known by the proof of ([11] Th. 6.2.2) that \( A^1(G) \subset A(G) \), where
\[
A(G) = \left\{ \hat{f} \mid f \in L^1(G) \right\}.
\]

Since \( \bigwedge^W_K(G) \subset A^1(G) \), then we have \( \bigwedge^W_K(G) \subset A^1(G) \subset A(G) \).

Again the Banach algebra \( A^p_{w,\omega}(G) \) is defined to be the set of functions \( f \in L^1_w(G) \) such that \( \hat{f} \in L^p_w(\hat{G}) \) with the norm
\[
\| f \|_{p,w,\omega} = \| f \|_{p,w} + \| \hat{f} \|_{p,\omega}, \quad 1 \leq p < \infty,
\]
where \( w \) and \( \omega \) are Beurling's weight functions on \( G \) and \( \hat{G} \), respectively \([4]\). It is known that if \( w \) satisfies (BD), then the regular maximal ideal
space of $L^1_w(G)$ is homeomorphic to the one of $A^p_{w,\infty}(G)$, ([5], Theorem 1.16). It is also known that if $W$ satisfies (B.D) then the regular maximal ideal space of $L^1_w(G)$ is the dual group $\hat{G}$ ([2], pp.15 and theorem 2.11). Then if $W$ satisfies (BD), the regular maximal ideal of $A^p_{w,\infty}(G)$ is the dual space $\hat{G}$.

2. THE SPACES $E^W(G)$ AND THEIR PROPERTIES

Let $G$ be a local compact abelian group, $K$ and $\hat{L}$ be the compact subsets of $G$ and $\hat{G}$, respectively. We define the vector space $E^W_{K,\hat{L}}(G)$ as the space of all function $u$ which can be represented as

$$u = \sum_{k=1}^{\infty} f_k * g_k , f_k \in K_k(G) , g_k \in L^1_w(G) , \hat{g}_k \in K_{\hat{L}}(\hat{G})$$

(1)

with

$$\sum_{k=1}^{\infty} \|f_k\|_\infty \cdot \|g_k\|_{L^1_w} < \infty$$

If one endows it with the norm

$$\|u\|_{E^W_{K,\hat{L}}(G)} = \inf \sum_{k=1}^{\infty} \|f_k\|_\infty \cdot \|g_k\|_{L^1_w} < \infty$$

then it is easy to see that $E^W_{K,\hat{L}}(G)$ becomes a Banach space under this norm, where the infimum is taken over all representations of $u$ as an element $E^W_{K,\hat{L}}(G)$ . The proof is similar to that of Guadry [4] and Larsen [5]). Now we define the vector space $E^W(G)$ to be

$$E^W(G) = \bigcup_{K,\hat{L}} E^W_{K,\hat{L}}(G)$$

(3)

together with the internal inductive limit topology of the Banach spaces $E^W_{K,\hat{L}}(G)$.

**Proposition 2.1.**

If $w$ satisfies the (B.D) condition then to every compact subset $\hat{K} \subset \hat{G}$ there is a constant $C_K > 0$ such that for every $f \in A^p_{w,\infty}(G)$ whose Fourier transform vanishes outside of $\hat{K}$ satisfies

$$\|f\|_{A^p_{w,\infty}} \leq C_K \cdot \|f\|_{L^1_w}$$

(1)

**Proof.** Since the (B.D) condition is satisfied, then for given any compact subset $\hat{K} \subset \hat{G}$ one can find a function $g \in A^p_{w,\infty}(G)$ such that
\( \hat{q}(x) = 1 \) for all \( x \in \hat{K} \). Take \( f \in A^p_{w,\omega}(G) \) satisfying \( \text{supp} \, \hat{F} \subset \hat{K} \). Hence we have \( f * g \in A^p_{w,\omega}(G) \) and
\[
\| f * g \|_{w,\omega}^p \lesssim \| f \|_{1,w} \cdot \| g \|_{w,\omega}^p
\]  
(2)

because \( A^p_{w,\omega}(G) \) is a module over \( f \in L^1_w(G) \), ([3]). If we set \( C^p = \| g \|_{w,\omega}(G) \) then find
\[
\| f * g \|_{w,\omega}^p \lesssim C^p \cdot \| f \|_{1,w}.
\]  
(3)

Because the hypothesis, \( \text{supp} \, \hat{F} \subset \hat{K} \) and \( \hat{g}(\hat{x}) = 1 \) over \( \hat{K} \), we write \( f * g = \hat{f} \cdot \hat{g} = \hat{f} \). Hence combining (2) and (3) we have
\[
\| f \|_{w,\omega}^p = \| f * g \|_{w,\omega}^p \lesssim C^p \cdot \| f \|_{1,w}.
\]  
(4)

**Lemma 2.2.** If \( w \) satisfies the (B.D) condition, then the norms \( \| \| \|_{1,w} \) and \( \| \|_{w,\omega}^p \) are equivalent on \( W^{w}_{K,L}(G) \).

**Proof.** It is easy to see that \( W^{w}_{K,L}(G) \subset A^p_{w,\omega}(G) \) by the Theorem 4.2. in [2]. Let \( f \in W^{w}_{K,L}(G) \) be given. Since \( \text{supp} \, \hat{F} \subset \hat{K} \), by the proposition 2.1, one can find a constant \( C^p_L > 0 \) such that
\[
\| f \|_{w,\omega}^p \lesssim C^p_L \cdot \| f \|_{1,w}.
\]

It is also known that
\[
\| f \|_{1,w} \lesssim \| f \|_{w,\omega}^p.
\]

Therefore these two norms are equivalent on \( W^{w}_{K,L}(G) \).

**Theorem 2.3.** If \( w \) satisfies (B.D) then

1) \( E^W(G) \) is continuously embedded into \( A^p_{w,\omega}(G) \).

2) \( E^W(G) \) is everywhere dense in \( W^{w}_{K,L}(G) \) with respect to the norms \( \| \|_{1,w} \) and \( \| \|_{w,\omega}^p \).

3) \( E^W(G) \) is everywhere dense in \( A^p_{w,\omega}(G) \).

**Proof.**

1) Let \( u \in E^W(G) \). Then \( u \in E^W_{k,L}(G) \) for a pair \( K, \hat{L} \), where \( K \) and \( \hat{L} \) are compact subsets of \( G \) and \( \hat{G} \), respectively. Then \( u \) can be represent as
\[ u = \sum_{k=1}^{\infty} f_k \ast g_k, f_k \in K_K(G), \hat{g} \in K_{\hat{L}}(\hat{G}), \]
with
\[ \sum_{k=1}^{\infty} \| f_k \|_\infty \ast \| g_k \|_{1,w} < \infty \]  
(1)

Since \( L^1_w(G) \) is a Banach convolution algebra then we write
\[ \| u \|_{1,w} \leq \sum_{k=1}^{\infty} \| f_k \ast g_k \|_{1,w} \leq \sum_{k=1}^{\infty} \| f_k \|_{1,w} \cdot \| g_k \|_{1,w} \]  
(2)

\[ \leq M \cdot \sum_{k=1}^{\infty} \| f_k \|_{\infty} \cdot \| g_k \|_{1,w} \]  
(3)

where \( M = \sup_{x \in K} |W(x)| \cdot \mu(K) \) and \( \mu(K) \) is the measure of \( K \). Also we have
\[ \| \psi_{p,\omega} \|_{p,\omega} = \left( \int \left| \sum_{k=1}^{\infty} \hat{f}_k \cdot \hat{g}_k \right|_p \leq \sum_{k=1}^{\infty} \left( \int \| f_k(x) \cdot \hat{g}_k(x) \|^p \omega^p(x)dx \right)^{\frac{1}{p}} \]  
(4)

\[ \leq \sum_{k=1}^{\infty} \| f_k \|_{\infty} \cdot \| g_k \|_{1,w} \cdot \left( \int \omega^p(x)dx \right)^{\frac{1}{p}} \leq \sum_{k=1}^{\infty} \| f_k \ast g_k \|_{1,w} \cdot \left( \int \omega^p(x)dx \right)^{\frac{1}{p}} \]  
(5)

\[ \leq \sum_{k=1}^{\infty} \| f_k \|_{\infty} \cdot \| g_k \|_{1,w} \cdot \left( \int \omega^p(x)dx \right)^{\frac{1}{p}} \cdot \mu(K) = N \sum_{k=1}^{\infty} \| f_k \|_{\infty} \cdot \| g_k \|_{1,w} \]  
(6)

where \( N = \left( \int \omega^p(x)dx \right)^{\frac{1}{p}} \cdot \mu(K) \)

If one uses (3) and (4) obtains that \( E^W(G) \subset A^p_{w,\omega}(G). \) Also by the Lemma 2.2 and (2), (4) the restriction of the identity map \( i \) from \( E^W(G) \) into \( A^p_{w,\omega}(G) \) to every subspace \( E^W_{K,\hat{L}}(G) \) is continuous. Hence \( i \) is a continuous embedding from \( E^W(G) \) into \( A^p_{w,\omega}(G). \)

2) It is easy to see the inclusion \( E^W(G) \subset \Lambda^W_K(G). \) For the proof of denseness of \( E^W(G) \) in \( \Lambda^W_K(G) \) with respect to the norm \( \| u \|_{1,w} \) take any function \( h \in \Lambda^W_K(G). \) Because the definition of \( \Lambda^W_K(G) \) there exists a compact subset \( \hat{L} \subset \hat{G} \) such that \( \hat{h} \in K_{\hat{L}}(\hat{G}). \) Since \( w \) has (B.D) condition then \( \Lambda^W_K(G) \subset A^p_{w,\omega}(G) \) has an approximate identity \( (e_{r,\omega})_{r \in I} \) bounded in \( L^1_w(G) \) with compactly supported Fourier transforms [2]. \( L^1_w(G) \) also has another approximate identity \( (u_{r,\beta})_{r \in I} \) with compactly supported [6]. Hence
\[ h \ast e^{\alpha}_* u = u \ast h \ast e^{\beta}_* e^{\beta}_* E^W(G), \]
for all \( \beta \in J \) and
\[ \| h \ast e^{\beta}_* u \ast h \|_{1,w} \leq \| h \ast e^{\alpha}_* u \ast h \ast e^{\alpha}_* e^\beta \|_{1,w} + \| h \ast e^{\alpha}_* h \|_{1,w} \rightarrow 0. \]

Also since by the Lemma 2.2, the norms \( \| \cdot \|_{1,w} \) and \( \| \cdot \|_{w,\omega}^p \) are equivalent on \( \wedge^W_{K,L}(G) \) for each pair \((K, L)\), then it is easy to see that \( E^W(G) \) is everywhere dense in \( \wedge^W_{K,L}(G) \) with respect to the norm \( \| \cdot \|_{w,\omega}^p \).

3) We know that \( A^p_{w,\omega}(G) \) has an approximate identity bounded in the norm \( L^1_w(G) \) ([2], Theorem 4.2). Using this approximate identity, a simple calculation shows that \( \wedge^W_{K,L}(G) \) is everywhere dense in \( A^p_{w,\omega}(G) \). If one combines this result with the first part of this theorem, observe that \( E^W(G) \) is everywhere dense in \( A^p_{w,\omega}(G) \).

**Proposition 2.4.** If \( 1 \leq p < \infty \) then

1) \( L^1_w(G) \times L^p_{\omega}(\widehat{G}) \) is a Banach space with the norm
\[ \| (f,g) \| = \| f \|_{1,w} + \| g \|_{p,\omega} \]
where \( (f,g) \in L^1_w(G) \times L^p_{\omega}(\widehat{G}) \).

2) \( A^p_{w,\omega}(G) \) is a closed subspace of the space \( L^1_w(G) \times L^p_{\omega}(\widehat{G}) \).

3) Every bounded linear functional \( F \) on \( A^p_{w,\omega}(G) \) is represented by the formula
\[ F(f) = \int_G f(x) \phi(x) dx + \int_G f(y) \psi(y) dy \]
where \( f \in A^p_{w,\omega}(G) \), \( (\phi, \psi) \in L^\infty_w(G) \times L^q_{\omega}(\widehat{G}) \)
and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** The proof of (1) is easy. For the proof of (2), define function \( \phi_p(f) = (f, f) \) from \( A^p_{w,\omega}(G) \) into \( L^1_w(G) \times L^p_{\omega}(\widehat{G}) \). \( \phi_p \) is an isometry and \( A^p_{w,\omega}(G) \hookrightarrow L^1_w(G) \times L^p_{\omega}(\widehat{G}) \). This proves part (2).

Since \( \frac{1}{p} + \frac{1}{q} = 1 \), then the topological dual of \( L^1_w(G) \times L^p_{\omega}(\widehat{G}) \) is isomorphic to \( L^{\infty}_w(G) \times L^{\infty}_{\omega}(\widehat{G}) \) and every continuous linear functional on \( L^1_w(G) \times L^p_{\omega}(\widehat{G}) \) is represented by
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\( F(f) = \int_G f(x) \phi(x) dx + \int_{\hat{G}} \hat{f}(y) \psi(y) dy \) (1)

\((\phi, \psi) \in L_{w}^{\infty}(G) \times L_{q}^{\infty}(\hat{G}).\) Because the fact (2) and by the Hahn Banach theorem, every continuous linear functional on \(A_{w,\infty}^{p}(G)\) is also represented by the formula (1).

**Proposition 2.5.** If \(\gamma \in (A_{w,\infty}^{p}(G))^{'}\) and \(f, g \in A_{w,\infty}^{p}(G)\), then we have

\[ \langle f * g, \gamma \rangle = \int_G f(y) \cdot \langle \tau_y g, \gamma \rangle dy , \]

where \(\tau_y\) is the translation operator defined by \(\tau_y g(x) = g(x-y)\).

**Proof.** By the proposition 2.4. we write

\[ \langle \hat{f} \hat{*} \hat{g}, \gamma \rangle = \int_G \langle \hat{f} \hat{*} g, \phi \rangle dx = \int_G \langle \hat{f} \hat{g}, \psi \rangle dx, \] (1)

where \((\phi, \psi) \in L_{w}^{\infty}(G) \times L_{q}^{\infty}(\hat{G})\) and \(\frac{1}{p} + \frac{1}{q} = 1\). A simple calculation shows that

\[ \int_G \langle \hat{f} \hat{g}, \phi \rangle dx = \int_G f(y) \langle \tau_y g, \phi \rangle dy \] (2)

and

\[ \int_G \langle \hat{f} \hat{g}, \psi \rangle dx = \int_G f(t) \langle \tau_t g, \psi \rangle dy \] (3)

If one combines these results obtains

\[ \langle \hat{f} \hat{*} \hat{g}, \gamma \rangle = \int_G f(t) \langle \tau_t g, \phi \rangle dt + \int_G f(t) \langle \tau_t g, \psi \rangle dt = \int_G f(t) \langle \tau_t g, \phi \rangle dt + \int_G f(t) \langle \tau_t g, \psi \rangle dt = \int_G f(t) \langle \tau_t g, \gamma \rangle dt. \]

**Proposition 2.6.** Let \(h \in \bigwedge_{w}^{K}(G)\). If \(w\) is symmetric and \(u \in E_{w}^{w}(G)\), then \(u \rightarrow \tilde{h} * u\) is a continuous function from \(E_{w}^{w}(G)\) into \(E_{w}^{w}(G)\), where

\(\tilde{h}(x) = h(-x)\).

**Proof.** Let \(u \in E_{w}^{w}(G)\). There is a pair \((K, \hat{L})\) such that \(f_k \in K_{K}(G)\), \(\hat{g}_k \in K_{\hat{L}}(\hat{G})\),

\[ u = \sum_{k=1}^{\infty} f_k * g_k \quad \text{and} \quad \sum_{k=1}^{\infty} \|f_k\|_{1,w} \cdot \|g_k\|_{1,w} < \infty . \] (1)

Since \(\tilde{h}, f_k \in L_{w}^{w}(G)\) then we have
\[
\tilde{h} \ast u = \sum_{k=1}^{\infty} f^*_{k} \ast (\tilde{h} \ast g_{k})
\]
and
\[
\sum_{k=1}^{\infty} \|f^*_{k}\|_{\infty} \|\tilde{h} \ast g_{k}\|_{1,w} \leq \|\tilde{h}\|_{1,w} \sum_{k=1}^{\infty} \|f^*_{k}\|_{\infty} \cdot \|g_{k}\|_{1,w} < \infty
\]  \(2\)

Hence \(\tilde{h} \ast u \in E^W(G)\). For the continuity, it is enough to show the restriction of the mapping \(u \rightarrow \tilde{h} \ast u\) to each \(E^W_{K,L}(G)\) is continuous. But this is immediate because if \(u_n - u_{K,L}^\wedge \rightarrow 0\) then we have
\[
\|\tilde{h} \ast u_n - \tilde{h} \ast u_{K,L}^\wedge\|_{1,w} \leq \|\tilde{h}\|_{1,w} \cdot \|u_n - u_{K,L}^\wedge\| \rightarrow 0.
\]  \(3\)

The proof of the following proposition is clear because of the Theorem 2.3. and Proposition 2.1.

**Proposition 2.7.** If \(w\) satisfies the (B.D) then we have \((A^p_{w,\omega}(G))' \subset (E^w(G))'\), where \((A^p_{w,\omega}(G))'\) and \((E^w(G))'\) are topological duals of \(A^p_{w,\omega}(G)\) and \(E^w(G)\) respectively.

**Definition 2.8.** Let \(f \in A^w_{K}(G), \sigma \in (E^w(G))'\) and \(w\) be a symmetric Beurling's weight. We are going to define the convolution \(\sigma \ast f\) to be
\[
\langle u, \sigma \ast f \rangle = \langle f \ast u, \sigma \rangle
\]  \(1\)

where \(u \in E^w(G)\). It is easily seen that \(1\) is well defined because the Proposition 2.6.

Let \(w\) be a symmetric weight and \(u \in (E^w(G))'\). Then the linear fractional \(\tilde{u} \in (E^w(G))^\prime\) is defined to be \(\langle u, \tilde{u} \rangle = \langle \tilde{u}, u \rangle\) for all \(u \in E^w(G)\).

### 3. Multipliers on the Space \(A^p_{w,\omega}(G)\).

**Definition 3.1.** A multipliers on \(A^p_{w,\omega}(G)\) is a bounded linear operator \(T\) on \(A^p_{w,\omega}(G)\) which commutes with translation operators, that is \(T\tau_s = \tau_s T\) for each \(s \in G\). The space of all multipliers on \(A^p_{w,\omega}(G)\) will be denoted by \(M(A^p_{w,\omega}(G))\).

**Proposition 3.1.** If \(T \in M(A^p_{w,\omega}(G))\), then \(T(f \ast g) = Tf \ast g\) for all \(f, g \in A^p_{w,\omega}(G)\).
Proof. Take any $T \in M(A^p_{w,\omega}(G))$, $f \in A^p_{w,\omega}(G)$ and $\gamma \in (A^p_{w,\omega}(G))^\prime$. It is easy to prove that the map $f \rightarrow \langle Tf, \gamma \rangle$ is a continuous linear functional on $A^p_{w,\omega}(G)$. Then there exists $\psi \in (A^p_{w,\omega}(G))^\prime$ such that $\langle f, \psi \rangle = \langle Tf, \gamma \rangle$ for all $f \in A^p_{w,\omega}(G)$. By the Proposition 2.5. one can write
\[
\langle Tf^g, \psi \rangle = \int_g (g(y)^{(T_y f, \gamma)} dy = \int_g (T_y f, \psi)^{y} dy = \langle f^g, \psi \rangle = \langle T(f^g), \gamma \rangle.
\]
Using the Hahn Banach theorem we obtain $Tf^g = T(f^g)$ for every $f, g \in A^p_{w,\omega}(G)$.

Theorem 3.2. Let $w$ be a symmetric weight on $G$ satisfying (B.D). If $T \in M(A^p_{w,\omega}(G))$, then there exists a unique continuous linear functional $\sigma \in (E^W(G))^\prime$ such that $Tf = \sigma \ast f$ for all $f \in \wedge_k(G)$.

Proof. If $u \in E^W_{k,l}(G)$ then one writes
\[
u = \sum_{k=1}^\infty f_k \ast g_k
\]
for some $f_k \in K_k(G)$ and $g_k \in L^1_w(G)$ satisfying $\hat{g}_k \in K^\wedge(G)$. By the Proposition 2.1. we have
\[
| (f_k \ast \mathcal{T}_k)(0) | \leq \| f_k \|_{\infty} \cdot \| \mathcal{T}_k \|_1 \leq \| f_k \|_{\infty} \cdot \| \mathcal{T}_k \|_{w,\omega}^p \leq C_{L^1} \cdot \| T \| \| f_k \|_{\infty} \cdot \| g_k \|_{1,w}.
\]
Hence the series
\[
\sum_{k=1}^\infty f_k \ast \mathcal{T}_k(0),
\]
converges uniformly. If we set
\[
u(u) = \sum_{k=1}^\infty f_k \ast \mathcal{T}_k(0),
\]
then it is easy to see that $\nu$ is well defined in the following means: If
\[
\sum_{k=1}^\infty f_k \ast g_k
\]
is a representation of 0 as an element of $E^W_{k,l}(G)$ then
\[ \sum_{k=1}^{\infty} f_k \ast Tg_k(0) = 0 \]

Using the formula (2) one obtains

\[ |u(u)| \leq C_L \|T\| \cdot |u|_{K,L} \]

for all \( u \in \mathcal{W}_k\). Therefore \( u \in (\mathcal{W}_k)' \). Hence we have \( \langle u, \tilde{u} \ast f \rangle = \langle \tilde{f} \ast u, \tilde{u} \rangle = \langle \tilde{u} \ast T f(0), \tilde{u} \rangle = \langle u, T f \rangle \) for all \( u \in \mathcal{W}_k \) and \( f \in \mathcal{W}_k \). That means \( T f = \tilde{u} \ast f \) for each \( f \in \mathcal{W}_k \). We set \( \sigma = \tilde{u} \).

Also since \( w \) satisfies (B.D), then there is a bounded approximate identity \( (e_\alpha) \) in \( L^1_w(G) \) ([2] Th. 4.2.). Let

\[ h = \sum_{k=1}^{\infty} f_k \ast g_k \in \mathcal{W}_k \]

be given. Then there exists a pair \( (K, \hat{L}) \) such that \( h \in \mathcal{W}_k \). Since

\[ |e_\alpha \ast g_k - g_k|_{L_1,w} \to 0, \]

using the equality

\[ |e_\alpha \ast h - h|_{K,L} = |\sum_{k=1}^{\infty} f_k \ast [g_k - g_k]|_{K,L} \]

\[ = \inf \sum |f|_{L_\infty} \cdot |e_\alpha \ast g_k - g_k|_{L_1,w}. \]

one easily shows that the set

\[ \left\{ f \ast h f \in \mathcal{W}_k(G), h \in \mathcal{W}_k(G) \right\} \]

is dense in \( \mathcal{W}_k(G) \).

Assume that \( s \) is not unique. Then there exists \( \sigma, \sigma' \in (\mathcal{W}_k(G))' \) such that \( T f = \sigma \ast f = \sigma' \ast f \). Hence we have \( \langle f \ast h, \sigma \rangle = \langle f \ast h, \sigma' \rangle \) for all \( f \in \mathcal{W}_k(G) \) and \( h \in \mathcal{W}_k(G) \). Using the denseness of (3) in \( \mathcal{W}_k(G) \) one obtains that \( \sigma = \sigma' \). That means \( \sigma \) is unique.

We denote by \( A^\omega \) the Banach algebra \( \{L^1_\omega(G)\} \) with its natural norm \( \|f\|_1^{\omega} \) [12].

**Proposition 3.3.** If \( w \) and \( \omega \) satisfy (B.D) then \( \mathcal{W}_k(G) \) is dense in \( A^\omega(G) \).
Proof. Since \( \omega \) satisfies (B.D) then \( \left(L_1^w(G)\right) \) has a bounded Approximate identity \( \left(u_j\right)_{j \in J} \) (shortly BAI) whose Fourier transforms have compact support ([3], Th. 4.2.). So, it is easily proved that the set \( A^\omega_c(G) = A^\omega(G) \cap K(G) \) is dense in \( A^\omega(G) \). Since \( W \) satisfies (B.D) then \( L^1_w(G) \) also has a BAI \( \left(e_\alpha\right)_{\alpha \in I} \) whose Fourier transforms have compact support. Suppose \( \hat{f} \in A^\omega_c(G) \). Then \( \left(e_\alpha * \hat{f}\right) \subset \Lambda^w(K) \) for all \( \alpha \in I \). Again because the regularity of \( \hat{f} \), given any compact subset \( K_0 \subset \hat{G} \) there exists \( g \in \Lambda^w(K) \) such that \( \hat{g}(x) = 1 \) for all \( x \in K_0 \). Therefore we obtain

\[
\left\| e_\alpha - 1 \right\|_{L^1_{1,\omega}} = \sup_{x \in K_0} \left| \hat{e}_\alpha(x) - 1 \right| \leq \left\| \hat{f} \right\|_{1,\omega} \rightarrow 0.
\]

(1)

we let \( C_0 = C + 1 \) where \( \left\| e_\alpha \right\| \leq C \), for all \( \alpha \in I \). Since \( \hat{f} \in A^\omega_c(G) \), then given \( \varepsilon > 0 \) there exists a compact subset \( K \subset \hat{G} \) such that

\[
\frac{1}{\hat{G}} \int_{\hat{G}} |f(x)| \omega(x) \, dx < \frac{\varepsilon}{2C_0}.
\]

(2)

Moreover, because the formula (1) there exists an \( \alpha_0 \in I \) such that if \( \alpha > \alpha_0 \) then

\[
\left\| e_\alpha - 1 \right\|_{L^1_{1,\omega}} = \sup_{x \in K} \left| \hat{e}_\alpha(x) - 1 \right| < \frac{\varepsilon}{2\left\| f \right\|_{1,\omega}}.
\]

(3)

Using (2) and (3) we have

\[
\left\| \hat{f} - e_\alpha * \hat{f} \right\|_{L^1_{1,\omega}} = \left\| f - e_\alpha * f \right\|_{1,\omega}
\]

\[
= \int_{\hat{G}} \left| f(x) - e_\alpha(x)f(x) \right| \omega(x) \, dx + \int_{K} \left| f(x) - e_\alpha(x)f(x) \right| \omega(x) \, dx
\]

\[
\leq \left( 1 + \left\| e_\alpha \right\|_{\infty} \right) \int_{\hat{G}} |f(x)| \omega(x) \, dx + \left\| f \right\|_{1,\omega} \left\| 1 - e_\alpha \right\|_{L^1_{1,\omega}}
\]

\[
\leq (1 + C) \int_{\hat{G}} |f(x)| \omega(x) \, dx + \left\| f \right\|_{1,\omega} \left\| 1 - e_\alpha \right\|_{L^1_{1,\omega}}
\]

\[
\leq C_0 \cdot \frac{\varepsilon}{2C_0} + \frac{\varepsilon}{2\left\| f \right\|_{1,\omega}} \cdot \left\| f \right\|_{1,\omega} = \varepsilon.
\]

Since \( A^\omega_c(G) \) is dense in \( A^\omega(G) \), then given any \( \hat{g} \in A^\omega(G) \) one can find \( f \in A^\omega_c(G) \) such that \( \left\| \hat{f} - \hat{g} \right\| < \varepsilon \). Then

\[
\left\| \hat{g} - e_\alpha * \hat{f} \right\|_{L^1_{1,\omega}} \leq \left\| \hat{g} - \hat{f} \right\|_{L^1_{1,\omega}} + \left\| \hat{f} - e_\alpha * \hat{f} \right\|_{L^1_{1,\omega}} < 2\varepsilon
\]

(4)

for all \( \alpha \geq \alpha \). This completes the proof.
Corollary 3.4. If \( w \) and \( \omega \) satisfy the conditions in Proposition 3.3, then \( \sigma^w_\infty(G) \) is dense in \( A(G) \).

**Proof.** Suppose \( \hat{g} \in A(G) \). Since \( K(G) \) is everywhere dense in \( L^1(G) \), then given any \( \varepsilon > 0 \) there exists \( h \in K(G) \subset L^1(G) \) such that
\[
\| g - h \|_A < \frac{\varepsilon}{2}.
\]
Hence by the Proposition 3.3, one can find \( \hat{k} \in \sigma^w_\infty(G) \) such that
\[
\| k - h \|_A < \frac{\varepsilon}{2}.
\]
Combining (1) and (2) we have
\[
\| \hat{g} - k \|_A < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
This proves our Corollary.

Now we recall that \( A(G) \) is a Banach algebra under pointwise multiplication operation with the norm \( \| f \|_A = \| f \|_1 \). Every continuous linear functional on \( A(G) \) is called a pseudomeasure.

**Proposition 3.5.** If \( A'(G) \) denotes the algebra of all pseudomeasures on \( G \), then \( A'(G) \subset (E^w(G))^\prime \).

**Proof.** Suppose that \( u \in E^w(G) \). Then there exists a pair \((K, \hat{L})\) of compact sets such that \( u \in E_{K,\hat{L}}(G) \). We also have
\[
\| u \|_A = \left\| \sum_{k=1}^{\infty} f_k \ast g_k \right\|_A \leq \sum_{k=1}^{\infty} \| f_k \ast g_k \|_A = \sum_{k=1}^{\infty} \left\| \hat{f}_k \ast \hat{g}_k \right\|_1 = \sum_{k=1}^{\infty} \left\| \hat{f}_k \right\|_1 \cdot \left\| \hat{g}_k \right\|_1.
\]
If one combines the inequality
\[
\| \hat{g}_1 \|_1 \leq \| \hat{g}_1 \|_1 \cdot \mu(K)
\]
with (1), obtains
\[
\| u \|_A \leq \sum_{k=1}^{\infty} \| f_k \|_1 \cdot \left\| \hat{g}_k \right\|_1 \cdot \mu(L) \cdot \mu(K)
\]
\[
\leq \sum_{k=1}^{\infty} \| f_k \|_1 \cdot \| g_k \|_1 \cdot \mu(L) \cdot \mu(K) \leq \sum_{k=1}^{\infty} \| f_k \|_1 \cdot \| g_k \|_1 \cdot \mu(L) \cdot \mu(K) < \infty.
\]
That means \( E^w(G) \subset A(G) \). Now if \( \sigma \in A'(G) \) and \( u \in E^w_{K,\hat{L}}(G) \) then we have
\[
\| u \|_A \leq \| \sigma \|_A \leq \| \sigma \| \cdot \sum_{k=1}^{\infty} \| f_k \|_1 \cdot \| g_k \|_1 \cdot \mu(L) \cdot \mu(K).
\]
Therefore
\[ |u, \sigma| \leq |\sigma| \cdot |u|_{k,L}^W \mu(L) \mu(K) \]

Since \( \sigma \) is continuous on every \( E^W_{k,L}(G) \) then \( \sigma \in (E^W(G)) \). This completes the proof.

**Theorem 3.6.** Assume that \( w \) and \( \omega \) satisfy (B.D) and \( W \) is symmetric. If \( T \in M(A^p_{w,\omega}(G)) \), then there exists a unique pseudo-measure \( \sigma \in A'(G) \) such that \( Tf = \sigma \ast f \) for all \( f \in A^p_{w,\omega}(G) \).

**Proof.** Suppose that \( T \in M(A^p_{w,\omega}(G)) \). By the proposition 3.1, we have \( T(f \ast g) = Tf \ast g \) for all \( f, g \in A^p_{w,\omega}(G) \), since \( A^p_{w,\omega}(G) \) is commutative it is easy to see that \( Tf \ast g = f \ast Tg \) for all \( f, g \in A^p_{w,\omega}(G) \). Then we write \( (Tf) \circ \hat{g} = \hat{f} \circ (Tg) \). Since \( W \) satisfies (B.D) then \( A^p_{w,\omega}(G) \) has approximate identities ([4], Theorem 4.2). Also it is known that \( A^p_{w,\omega}(G) \) is a Banach convolution algebra ([4], Theorem 2.1). Hence \( A^p_{w,\omega}(G) \) is a commutative Banach algebra without order (i.e if for all \( f \in A^p_{w,\omega}(G) \), \( f \ast A^p_{w,\omega}(G) = 0 \) then \( f = 0 \)). Again since \( W \) satisfies (B.D) then the regular maximal ideal space of \( L^1_w(G) \) is the dual group \( \hat{G} \) ([2], pp.15 and Theorem 2.11). It is also known that in the case \( W \) satisfies (B.D) condition the regular maximal ideal space of \( L^1_w(G) \) is homeomorphic to the one of \( A^p_{w,\omega}(G) \), ([5], Th. 1.16), which implies that the regular maximal ideal space of \( A^p_{w,\omega}(G) \) is the dual space \( \hat{G} \). Then there exists a unique bounded continuous function \( \Phi \) on \( \hat{G} \) such that \( (Tf) \circ \hat{g} = \Phi(y) \circ \hat{g}(y) \) for all \( g \in A^p_{w,\omega}(G) \) by the Theorem 1.2.2. in [11]. If \( f \in \wedge^W_K(G) \) then \( Tf \in L^1_w(G) \) and \( (Tf) \circ \hat{g} = \Phi \circ \hat{g} \in \hat{K}(G) \). Therefore \( \wedge^W_K(G) \) is invariant under \( T \). Since every element of \( \wedge^W_K(G) \) is continuous (see introduction) then we can define a linear functional on \( \wedge^W_K(G) \) as \( L(f) = Tf(0) \) for all \( f \in \wedge^W_K(G) \). Also we write,

\[ |L(f)| = |Tf(0)| \leq \| Tf \|_{w} \tag{1} \]

Since \( Tf \in \wedge^W_K(G) \subset A(G) \) then there exists \( g \in L^1(G) \) such that \( \hat{g} = Tf \). If one uses the inequalities \( \hat{g} = \tilde{g} \) and \( ||\tilde{g}||_1 = ||\hat{g}||_1 \) writes

\[ ||\tilde{g}||_1 = ||\hat{g}||_1 = ||\hat{g}||_1 \tag{2} \]
where \( \hat{g}(x) = g(-x) \). Now if we combine (1) and (2) obtain

\[
|L(f)| \leq \|Tf\|_\infty = \|\hat{\Phi} \hat{f}\|_\infty \leq \|\Phi\|_1 \cdot \|f\|_1 = \|\Phi\|_\infty \cdot \|\hat{\Phi}\|_1 = \|\Phi\|_\infty \cdot \|f\|_A.
\]

(3)

Thus \( L \) is a continuous linear functional on \( \Lambda^W_K(G) \). Since \( \Lambda^W_K(G) \) is dense in \( A(G) \) by the Corollary 3.4., then \( L \) can be extended uniquely as a continuos linear functional on \( A(G) \). Hence there exists a unique pseudo-measure \( \sigma \) such that

\[
L(f) = Tf(0) = \langle f, \sigma \rangle
\]

(4)

for all \( f \in \Lambda^W_K(G) \). Then \( Tf = \sigma * f \) for all \( f \in \Lambda^W_K(G) \). An examination proof of Theorem 3.2 and proposition 3.5 show that \( \sigma \) is a pseudo measure and is unique. Hence to complete the proof of this theorem it remains to show that \( Tf = \sigma * f \) holds for all \( f \in A^p_{w,\omega}(G) \). Let \( f \) be any element of \( A^p_{w,\omega}(G) \). If \( (e_{\alpha})_{\alpha \in I} \) is a bounded approximate identity for \( A^p_{w,\omega}(G) \) chosen from \( \Lambda^W_K(G) \) ([4], Th. 4.2) then for each \( f \in A^p_{w,\omega}(G) \) the net \( (e_{\alpha} * f) \) is Cauchy net in \( \Lambda^W_K(G) \) and since \( T(e_{\alpha} * f) = \sigma * (e_{\alpha} * f) \), we have

\[
|\sigma * (e_{\alpha} * f) - \sigma * (e_{\beta} * f)|_{W,\omega}^p
\leq \|T(e_{\alpha} * f) - T(e_{\beta} * f)|_{W,\omega}^p \leq \|e_{\alpha} * f - e_{\beta} * f||_{W,\omega}^p
\]

(5)

which implies that \( (\sigma * (e_{\alpha} * f))_{\alpha \in I} \) is a Cauchy net in \( A^p_{w,\omega}(G) \) and converges to a function \( F \in A^p_{w,\omega}(G) \). That means

\[
|F - \sigma * (e_{\alpha} * f)|_{W,\omega}^p \to 0.
\]

(6)

Again it is clear that \( \sigma * f \in A'(G) \) because \( f \in L^1(G) \) and \( \sigma \in A'(G) \).

If we use (6) and the relation

\[
\|\hat{F} - \hat{\sigma}\|_\infty \leq \|\hat{F} - \hat{\sigma} \hat{\sigma}^{\hat{\alpha}} \|_\infty + \|\hat{\sigma} \hat{\sigma}^{\hat{\alpha}} - \hat{\sigma}\|_\infty
\leq \|F - \sigma * (e_{\alpha} * f)\|_1 + \|\sigma^{\hat{\alpha}}\|_1 \cdot |e_{\alpha} * f - f|_1
\leq \|F - \sigma * (e_{\alpha} * f)\|_{W,\omega}^p + \|\sigma\|_{W,\omega} \cdot |e_{\sigma} * f - f|_{W,\omega}^p
\]

(7)
find that $\hat{F} = \hat{\sigma} \cdot \hat{f}$. From the inversion theorem we write $F = \sigma \ast f$. Also we have

$$\| T\hat{f} - \sigma \ast (e_\alpha \ast f) \|_{L^p_{w,0}} = \| T\hat{f} - T(e_\alpha \ast f) \|_{L^p_{w,0}}$$

$$\leq \| \hat{f} - e_\alpha \ast d_{w,0}^p \| \rightarrow 0.$$  \hspace{1cm} (8)

Consequently it follows from (6), (7) and (8) that $Tf = F = \sigma \ast f$ for all $f \in A^p_{w,0}(G)$. This completes the proof.

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