THE BOUNDS FOR PERRON ROOTS OF GCD, GMM, AND AMM MATRICES

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ABSTRACT

In this paper we define the greatest common divisor matrix (or GCD), the geometric mean matrix (or GMM) and the arithmetic mean matrix (or AMM) on the set \( E = \{1, 2, 3, \ldots, n\} \) and we obtain the bounds for the Perron root of these matrices.

INTRODUCTION AND MAIN RESULTS

Definition 1. Let \( S = \{x_1, x_2, \ldots, x_n\} \) be a finite ordered set of distinct positive integers. The greatest common divisor matrix (GDC) defined on \( S \) is given by

\[
\begin{bmatrix}
(x_1, x_1) & (x_1, x_2) & \cdots & (x_1, x_n) \\
(x_2, x_1) & (x_2, x_2) & \cdots & (x_2, x_n) \\
\vdots & \vdots & \ddots & \vdots \\
(x_n, x_1) & (x_n, x_2) & \cdots & (x_n, x_n)
\end{bmatrix}
\]

and is denoted by \([S]_{gcd}\). In order words, for \( S = \{x_1, x_2, \ldots, x_n\} \),

\([S]_{gcd} = (s_{ij})_{n \times n}\), where \( s_{ij} = (x_i, x_j) = \gcd(x_i, x_j) \).

Definition 2. \( S = \{x_1, x_2, \ldots, x_n\} \) be a finite ordered set of distinct positive integers. The geometric mean matrix (GMM) defined on \( S \) is given by

\[
\begin{bmatrix}
\sqrt{x_1, x_1} & \sqrt{x_1, x_2} & \cdots & \sqrt{x_1, x_n} \\
\sqrt{x_2, x_1} & \sqrt{x_2, x_2} & \cdots & \sqrt{x_2, x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\sqrt{x_n, x_1} & \sqrt{x_n, x_2} & \cdots & \sqrt{x_n, x_n}
\end{bmatrix}
\]

and is denoted by \([S]_{gmm}\). In other words, for \( S = \{x_1, x_2, \ldots, x_n\} \),

\([S]_{gmm} = (g_{ij})_{n \times n}\), where \( g_{ij} = \sqrt{x_i, x_j} \).
Definition 3. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite ordered set of distinct positive integers. The arithmetic mean matrix (AMM) defined on $S$ is given by

$$
\begin{bmatrix}
\frac{x_1 + x_2}{2} & \frac{x_1 + x_3}{2} & \cdots & \frac{x_1 + x_n}{2} \\
\frac{x_2 + x_1}{2} & \frac{x_2 + x_3}{2} & \cdots & \frac{x_2 + x_n}{2} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{x_n + x_1}{2} & \frac{x_n + x_2}{2} & \cdots & \frac{x_n + x_n}{2}
\end{bmatrix}
$$

and is denoted by $[S]_{\text{amm}}$. In other words, for $S = \{x_1, x_2, \ldots, x_n\}$, 

$$
[S]_{\text{amm}} = (a_{ij})_{n \times n}, \quad \text{where} \quad a_{ij} = \frac{x_i + x_j}{2}.
$$

Theorem 1 [1]. Let $A, B \in M_n$. If $0 \leq A \leq B$, then

$$
\rho(A) \leq \rho(B),
$$

where $\rho(.)$ denotes spectral radius i.e.,

$$
\rho(A) = \max \{|\lambda_1(A)|\}.
$$

Definition 4. A real $n$-square matrix $A = (a_{ij})$ is called nonnegative, if $a_{ij} \geq 0$ for $i, j = 1, 2, \ldots, n$. We write $A \geq 0$.

Definition 5. Let $A$ be a square nonnegative matrix. Then a nonnegative eigenvalue $r(A)$ which is not less than the absolute value of any other eigenvalue of $A$ is called Perron root.

Theorem 2. If $[S]_{\text{gcd}}$, $[S]_{\text{gmm}}$ and $[S]_{\text{amm}}$ denote GCD, GMM and AMM matrices on $S = \{x_1, x_2, \ldots, x_n\}$, respectively, then

$$
r([S]_{\text{gcd}}) < r([S]_{\text{gmm}}) < r([S]_{\text{amm}})
$$

Proof. In the following inequality is always true:

$$
(x_i, x_j) \leq \sqrt{x_i x_j} \leq \frac{x_i + x_j}{2} \quad (1)
$$

the equality hold if and only if $x_i = x_j$. So from the inequality (1) we have

$$
[S]_{\text{gcd}} \leq [S]_{\text{gmm}} \leq [S]_{\text{amm}}.
$$
Thus considering Theorem 1, it follows that the proof of theorem, is complete

**Theorem 3.** If $A$ is an $n \times n$ symmetric matrix, then

$$r(A) \geq \frac{E^T A e}{e^T e},$$

(2)

where $r(A)$ denotes Perron root of $A$ and $e^T = (1, 1, ..., 1)$.

**Proof.** We recall first the classical lower Frobenius bound of the Perron root an $n \times n$ nonnegative matrix $A$ (see, e.g., [2]),

$$r(A) \geq \min_i P_i,$$

(3)

where $P_i = P_i(A) = \sum a_{ij}$ is the $i$-th row sum of $A$. Obviously when $A$ is symmetric [since the Rayleigh quotient is a lower bound for $r(A)$] the bound (3) can be improved as follows:

$$r(A) \geq \frac{E^T A e}{e^T e} = \frac{1}{n} \sum_{i=1}^{n} P_i$$

Thus the proof is complete.

**Remark.** Unfortunately, for unsymmetric matrix $A$, the bound (2) can be wrong. Indeed, for

$$A = \begin{bmatrix} 2 & 2 \\ a & 2 \end{bmatrix}, a > 0$$

we have

$$\frac{E^T A e}{e^T e} = \frac{6 + a}{2}$$

On the other hand since $r(A) = 2 + \sqrt{2a}$, the lower bound (2) is valid if and only if

$$2 + \sqrt{2a} \geq \frac{6 + a}{2}$$

i.e., if $a = 2$ or, in other words, if $A$ is symmetric.

**Theorem 4.** Let $[E]_{amn}$ be arithmetic mean matrix (AMM) on $E = \{1, 2, 3, ..., n\}$. Then

$$\frac{E^T [E]_{amn} e}{e^T e} = \frac{n(n + 1)}{2}$$

where $e = (1, 1, ..., 1)^T$. 

**Proof.** It is easily seen that $e^T e = n$. On the other hand considering
\[ \sum_{i=1}^{n} x_i = \frac{n(n + 1)}{2} \]
we have
\[ e^T [E]_{gmm} e = \sum_{i,j=1}^{n} \frac{x_i + x_j}{2} = \sum_{i,j=1}^{n} \frac{x_i}{2} + \sum_{i,j=1}^{n} \frac{x_j}{2} \]
\[ = \sum_{i=1}^{n} \frac{x_i}{2} \sum_{j=1}^{n} 1 + \sum_{j=1}^{n} \frac{x_j}{2} \sum_{i=1}^{n} 1 \]
\[ = \frac{n^2(n + 1)}{2} \]
Consequently since $e^T e = n$, we write
\[ \frac{e^T [E]_{gmm} e}{e^T e} = \frac{n(n + 1)}{2} \]
Thus the proof is complete.

**Lemma 1.** Let $[E]_{gmm}$ be geometric mean matrix (GMM) on $E = \{1, 2, 3, ..., n\}$. Then

(i) $\det([E]_{gmm}) = 0$

(ii) $\text{rank}([E]_{gmm}) = 1$.

**Proof.** If $r_1, r_2, ..., r_n$ denote the rows of the matrix $[E]_{gmm}$, then we have
\[ r_k = \sqrt{k} \ r_1 \quad (k = 2, 3, ..., n) \]  
(4)
So by the properties of the determinants it follows that (i). on the other hand by the elementary row operations it follows that (ii).

Thus lemma is proved.

**Theorem 5.** Let $[E]_{gmm}$ be geometric mean matrix (GMM) on $E = \{1, 2, 3, ..., n\}$. Then
\[ r([E]_{gmm}) = \frac{n(n + 1)}{2} \]
where $r(.)$ denotes Perron root.
Proof. If \( \alpha_s \) is the sum of all principal minors of order \( s \) of \([E]_{gmm}\), \( 1 \leq s \leq n \), then we have

\[
\det(\lambda I - [E]_{gmm}) = \lambda^n - \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \ldots + (-1)^n \alpha_n.
\]

In particular, we note that

\[
\alpha_1 = \sum_{i=1}^n x_i = \frac{n(n+1)}{2} \quad \text{and} \quad \alpha_n = \det([E]_{gmm}).
\]

So by Lemma 1. (i) we have \( \alpha_n = 0 \). On the other hand by the Lemma 1. (ii) we write

\[
\alpha_2 = \alpha_3 = \ldots = \alpha_n = 0.
\]

Thus we obtain

\[
\lambda^n - \frac{n(n+1)}{2} \lambda^{n-1} = 0
\]

or

\[
\lambda^{n-1} \left( \lambda - \frac{n(n+1)}{2} \right) = 0.
\]

Therefore the eigenvalues of the matrix \([E]_{gmm}\) are \( \lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} = 0 \) and

\[
\lambda_n = r(A) = \frac{n(n+1)}{2}.
\]

Thus the theorem is proved.

Theorem 6. Let \( S = \{x_1, x_2, \ldots, x_n\} \) be an factor-closed set, and let \([S]_{gcd}\) be the GCD matrix defined on \( S \). Then

\[
\det([S]_{gcd}) = \varphi(x_1) \varphi(x_2) \ldots \varphi(x_n),
\]

where \( \varphi(.) \) denotes Euler's totient function.

Corollary 1. If \([E]_{gcd}\) is the GCD matrix defined on \( E = \{1, 2, 3, \ldots, n\} \), then

\[
\det([E]_{gcd}) = \varphi(1) \varphi(2) \ldots \varphi(n).
\]

Proof. Since the set \( E = \{1, 2, 3, \ldots, n\} \) is factor-closed, the proof is immediately seen by Theorem 6.
Theorem 7. If \([E]_{\gcd}\) is the GCD matrix defined on \(E = \{1, 2, 3, \ldots, n\}\) then
\[
r\left([E]_{\gcd}\right) \geq \left[\prod_{i=1}^{n} \varphi(i)\right]^{1/n}
\]
where \(r(.)\) denotes Perron root and \(\varphi(.)\) denotes Euler's totient function.

Proof. If \(\lambda_i (i = 1, 2, \ldots, n)\) are eigenvalues of the matrix \([E]_{\gcd}\), then we have
\[
\det([E]_{\gcd}) = \prod_{i=1}^{n} \lambda_i \leq \prod_{i=1}^{n} r([E]_{\gcd}) = r([E]_{\gcd})^n.
\]

On the other hand by the Corollary 1., we write
\[
\varphi(1) \varphi(2) \ldots \varphi(n) \leq r([E]_{\gcd})^n
\]
or
\[
\left[\prod_{i=1}^{n} \varphi(i)\right]^{1/n} \leq r([E]_{\gcd})
\]
Thus the proof is complete.

NUMERICAL EXAMPLES

Example 1. Consider the set \(E = \{1, 2, 3\}\). Then we write
\[
[E]_{anm} = \begin{bmatrix}
1 & \frac{3}{2} & 2 \\
\frac{3}{2} & 2 & \frac{5}{2} \\
2 & \frac{5}{2} & 3
\end{bmatrix}
\]
and we find
\[
r([E]_{anm}) = 3 + \frac{1}{2} \sqrt[4]{2} \approx 6.24.
\]
Indeed, since \(\frac{n(n + 1)}{2} = 6\), we obtain \(6.24 \geq 6\).

Similarly for \(n = 4\) \(r([E]_{anm}) = 10.47 \geq 10\)
for \(n = 5\) \(r([E]_{anm}) = 15.79 \geq 15\)

etc.
Example 2. For $E = \{1, 2, 3\}$, since

$$[E]_{gmm} = \begin{bmatrix} 1 & \sqrt{2} & \sqrt{3} \\ \sqrt{2} & 2 & \sqrt{6} \\ \sqrt{3} & \sqrt{6} & 3 \end{bmatrix}$$

we obtain $r([E]_{gmm}) = 6$.

Similarly

- for $n = 4$, $r([E]_{gmm}) = 10$
- for $n = 5$, $r([E]_{gmm}) = 15$

etc.

REFERENCES

