POLYNOMIAL SOLUTIONS FOR A CLASS OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, polynomial solutions are given for a class of linear non-homogeneous singular partial differential equations of the second order. At the end of this paper, polynomial solutions are given for an iterated equation with order $2p$ which is obtained by applying the operator belonging to the same equation consecutively.

INTRODUCTION

Consider the following linear nonhomogeneous singular partial differential equation,

$$Lu = \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial y^2} + \alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} = q(x,y).$$

where $b$, $\alpha$ and $\beta$ are any real constants and $q$ is a polynomial in $\mathbb{R}^2$. Clearly the equation (1) includes some of the well-known classical equations such as the Laplace equation, the Poisson equation, the axially symmetric potential equation and the wave equation. To obtain a particular solution for the equation (1) in the case of $q(x,y) \neq 0$ is an important problem. From the theory of linear equations it is known that if we have a particular solution of the equation $Lu=q$ and we know the general solution of the equation $Lu=0$, then we can obtain the general solution of the equation $Lu=q$. In the equation (1), if $g(x,y) \equiv 0$ the polynomial solutions are given in [2,4]. In this paper, polynomial solutions are given for the equation (1) and polynomial solutions are given for the iterated equation $L^p(u) = 0$ for $p \geq 1$. The iterated operator $L^p$ is defined
by the relation.

\[ L^{s+1}(u) = L[L^s(u)] \quad s = 1, \ldots, p-1 \]

2. POLYNOMIAL SOLUTIONS FOR THE EQUATION (1)

In general a polynomial \( q(x,y) \) may be written in the form

\[ q(x,y) = \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} x^i y^j \; ; \; M, N \in \mathbb{N}. \]  \hspace{1cm} (2)

If \( q_{ij} = x^i y^j \) \( 0 \leq i \leq M, 0 \leq j \leq N \), then we can write \( q(x,y) = \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} q_{ij} \).

By the principle of superposition, it is known that if \( L p_{ij} = q_{ij} \), then we obtain

\[ L \left( \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} p_{ij} \right) = \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} L(p_{ij}) = q(x,y). \]

Hence, it is clear that for obtaining a particular solution \( p \) of \( L u = q \), it will be enough to find particular solutions \( u = p_{ij} \) satisfying

\[ L u = x^i y^j \quad i, j \in \mathbb{N} \]  \hspace{1cm} (3)

Thus, \( \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} p_{ij} = p \) becomes the required particular solution of the equation \( L u = q \). We explain below how the polynomial solutions are obtained when the typical terms on the right-hand side of the equation (3) are of the form \( x^i y^{2j}, x^{2i} y^j, x^{2j} y^i \).

**Theorem 1.** Let \( 0 \leq i \leq M, 0 \leq j \leq N \); \( M, N \in \mathbb{N} \) be nonnegative integers. Then the equation

\[ L u = \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial y^2} + \alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} = x^i y^{2j} \]  \hspace{1cm} (4)

has a polynomial solution

\[ p = \frac{1}{(i+2)[(i+1)+\alpha]} x^{i+2} y^{2j} + \sum_{s=2}^{i+1} b_{2s} x^{i+2s} y^{2j-2s+2} \]  \hspace{1cm} (5)

where

\[ b_{2s} = (-1)^{s-1} \frac{2j(2j-2)\ldots(2j-2s+4)(2j-1)b+\beta)\ldots(2j-2s+3)b+\beta]}{(i+2)\ldots(i+2s)[(i+1)+\alpha)\ldots[(i+2s-1)+\alpha]} \]  \hspace{1cm} (6)

\( s = 2, \ldots, j+1 \) and for \( s = 1, \ldots, j+1, \alpha \neq -(i+2s-1). \)
Proof. Because of the property of the operator L, a particular solution \( p(x,y) \) of the equation (4) should be a polynomial consisting of the terms of degree \((i+2j+2)\). Thus \( p(x,y) \) can be chosen as

\[
p = b_2 x^{i+2j+2} + b_4 x^{i+4j+2} y^{2j+2} + \ldots + b_{2s} x^{i+2sj+2j+2s+2} y^{2j+2s+2} \geq 0
\]

(7)

Now we calculated \( Lp \) from (4) and (7)

\[
Lp = (i+2)[(i+1) + \alpha] b_2 x^{i+2j+2} + [(i+4)(i+3) + \alpha] b_4 x^{i+4j+2} y^{2j+2} + \ldots + [(i+6)(i+5) + \alpha] b_{2s} x^{i+2sj+2j+2s+2} y^{2j+2s+2}
+ (2j-2s+2) [(2j-2s+1) + \alpha] b_{2s+2} x^{i+2s} y^{2j+2s+2} y^{2j+2s+2} = x^i y^j.
\]

Equating the coefficients of similar terms on both sides of the above identity, the following relations;

\[
b_2 = \frac{1}{(i+2)(i+1)+\alpha}
\]

is obtained from \((i+2)[(i+1)+\alpha] b_2 = 1\). Similarly, the other coefficients have the following forms.

\[
b_4 = -\frac{2j[(2j-1)b+\beta]}{(i+4)(i+3)+\alpha} b_2,
\]

\[
b_6 = -\frac{(2j-2)((2j-3)b+\beta)}{(i+6)(i+5)+\alpha} b_4,
\]

\[
b_{2s} = -\frac{(2j-2s+2)(2j-2s+3)b+\beta}{(i+2s)(i+2s-1)+\alpha} b_{2s-2}.
\]

By multiplying them side by side and writing the value of \( b_2 \), we obtain \( b_{2s} \) as defined in (6). Hence, we obtain \( p(x,y) \) as given in (5).

Theorem 2. Let \(0 \leq i \leq M, 0 \leq j \leq N; M, N \in N\) be nonnegative integers. Then the equation

\[
Lu = \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial y^2} + \frac{\alpha}{x} \frac{\partial u}{\partial x} + \frac{\beta}{y} \frac{\partial u}{\partial y} = x^{2i} y^j
\]

has a polynomial solution

\[
p = \frac{1}{(j+2)(i+1)+\beta} x^{2i} y^{i+2} + \sum_{s=2}^{i+1} b_{2s} x^{2i-2s} y^{2s+2} y^{2s+2}
\]

(8)

where

\[
b_{2s} = (-1)^{s-1} \frac{2i(2i-2)...(2i-2s+4)(2i-1+\alpha)...[2i-2s+3+\alpha]}{(j+2)...(i+2s)(i+1)+\beta}...(j+2s-1)+\beta
\]
s = 2, ... , i+1 and for s = 1, ... , i+1 \[ \beta \neq -(j+2s-1)b. \]

The proof is very similar to the proof of Theorem 1; so we shall not give it here. On the other hand, if the right-hand side of the equation (3) is of the form \( x^{2i}y^{2j} \), replacing \( i \) by \( 2i \) in (5) or \( j \) by \( 2j \) in (8), we simply obtain polynomial solutions of \( Lu = x^{2i}y^{2j} \). Hence, if typical terms in \( q \) are of the forms \( x^{i}y^{j} \), \( x^{2i}y^{2j} \), \( x^{2i}y^{2j} \), we obtain polynomial solution of the equation (1).

**Special Cases.** If \( q \) is a polynomial which is odd with respect to the variables \( x \) and \( y \) in its terms, then the equation (1) has polynomial solutions in some special cases. Namely, if the typical term is of the form \( x^{2n+1}y^{2m+1} \) in \( q \) (\( n,m \geq 1 \)), in the following special cases, we find polynomial solutions of the equation

\[
Lu = x^{2n+1}y^{2m+1} \quad (9)
\]

If we choose polynomial solution

\[
p = Ax^{2n+3}y^{2m+1} + Bx^{2n+1}y^{2m+3} \quad (10)
\]

for the equation (9) by applying the operator \( L \) in (9) to this function, we obtain

\[
L(p) = A(2m+1)[2mb+\beta]) \cdot x^{2n+3}y^{2m+1} + A(2n+3)[2(m+1)+\alpha]+B(2m+3)[2(m+1)b+\beta]) \cdot x^{2n+1}y^{2m+1} + B(2n+1)(2n+\alpha)x^{2n-1}y^{2m+1} = x^{2n+1}y^{2m+1}.
\]

From this identity, we obtain

\[
A(2m+1)[2mb+\beta] = 0 \\
B(2n+1)(2n+\alpha) = 0 \\
A(2n+3)[2(n+1)+\alpha]+B(2m+3)[2(m+1)b+\beta] = 1
\]

We can write the following special cases here.

**i.** The equation (9) has a polynomial solution

\[
p = \frac{1}{(2m+3)[2(m+1)b+\beta]} x^{2n+1}y^{2m+3} \quad \text{for } \alpha = -2n
\]

**ii.** The equation (9) has a polynomial solution

\[
p = \frac{1}{(2n+3)[2(n+1)+\alpha]} x^{2n+3}y^{2m+1} \quad \text{for } \beta = -2mb
\]
iii. If $A$ and $B$ are nonzero arbitrary constants which satisfy the equality $A(2n+3)[2(n+1)+\alpha]+B(2m+3)[2(m+1)b+\beta] = 1$, the equation (9) has a polynomial solution $p = Ax^{2n+3}y^{2m+1} + Bx^{2n+1}y^{2m+3}$ for $\alpha = -2n$, $\beta = -2mb$.

3. POLYNOMIAL SOLUTIONS FOR THE EQUATION $L^p(u) = 0$

The formula for the operator $L$ is easily derived.

$$L(fg) = gL(f) + 2\left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + b \frac{\partial f}{\partial y} \frac{\partial g}{\partial y}\right) + fL(g)$$  \hspace{1cm} (11)

In particular, if $f$ is replaced by $x^k$ ($k \in \mathbb{R}$) in (11), we obtain

$$L(x^kg) = k(k-1+\alpha)x^{k-2}g + 2kx^{k-1}\frac{\partial g}{\partial x} + x^kL(g).$$  \hspace{1cm} (12)

**Lemma 1.** Let $T^* = x \frac{\partial}{\partial x}$ and if $Lg = 0$, then

$$L(x^kg) = x^{k-2}k(k-1+\alpha+2T^*)g$$  \hspace{1cm} (13)

**Proof.** The proof is obvious from (12).

**Lemma 2.** Let $L_x = \frac{\partial^2}{\partial x^2} + \frac{\alpha}{x} \frac{\partial}{\partial x}$ and $A_j, B_j, C_j, D_j$ be real constants, then the functions

$$u_j(x,y) = A_jx^{1-\alpha} + B_jy^{1-\beta} + C_jx^{1-\alpha}y^{1-\beta} + D_j$$

are solutions of both the equations $L(u_j) = 0$ and $L_x(u_j) = 0$.

**Proof.** By applying the operator $L$ and $L_x$ to this function $u_j$ we simply see that $L(u_j) = 0$ and $L_x(u_j) = 0$.

**Lemma 3.** If $g$ is of the form (14), then

$$L^p(x^kg) = x^{k-2p} \prod_{j=0}^{p} (k - 2j)[k - 1 + \alpha + 2T^* - 2j]g.$$  \hspace{1cm} (15)

**Proof.** We prove this by the method of induction. Let $T = k-1+\alpha+2T^*$. From (13), we write $L(x^kg) = kx^{k-2}(Tg)$. Applying the operator $L$ on both sides of this equality and using (12), we obtain

$$L^2(x^kg) = kL[x^{k-2}Tg] = k \left( x^{k-4}(k - 2)[(k - 2) - 1 + \alpha]Tg + 2(k - 2) x^{k-2} \frac{\partial Tg}{\partial x} \right) + kx^{k-2}L(Tg).$$

On the other hand, by direct calculation, it can be shown that

$$LT^* = T^*L + 2L_x$$  \hspace{1cm} (17)
\[ LT = (k-1+\alpha)L + 2T^*L + 4L_x \]  

(18)

Here, let \( L_x \) be the same as in Lemma 2. From (18), we have \( L(Tg) = 0 \). In (16), by making use of \( L(Tg) = 0 \), we obtain

\[
L^2(x^kg) = k(k-2)x^{k-4}[(k-3)+\alpha+2T^*]Tg.
\]

Now, first assume that the equality (15) is true for \( p \) and show that it is true for \( p+1 \). Applying the operator \( L \) on both sides of the equality (15) and using (12), we obtain

\[
L^{p+1}(x^kg) = L \left( \sum_{j=0}^{p-1} x^{k-2(p+j)} \prod_{j=0}^{p-1} (k-2j)[T-2j]T \right) g
\]

\[
= (k-2p)[(k-2p-1)+\alpha] x^{k-2(p+1)} \prod_{j=0}^{p-1} (k-2j)[T-2j]T g
\]

\[
+ 2(k-2p) x^{k-2(p+1)} x \frac{\partial}{\partial x} \prod_{j=0}^{p-1} (k-2j)[T-2j]T g
\]

\[
+ x^{k-2p} L \left( \prod_{j=0}^{p-1} (k-2j)[T-2j]T g \right)
\]

By making use of \( L(Tg) = 0 \) we see that

\[
L \left( \prod_{j=0}^{p-1} (k-2j)[T-2j]T g \right) = 0.
\]

Hence, we obtain

\[
L^{p+1}(x^kg) = x^{k-2(p+1)} \prod_{j=0}^{p}(k-2j)[k-1+\alpha+2T^*-2j]T g.
\]

This completes the proof.

If \( f \) is replaced by \( y^k \) \((k \in R)\) in (11), we give the following Lemma. Its proof is very similar to the proof of Lemma 3; so we shall give it here without proof.

**Lemma 4.** If \( g \) is of the form of (14), then

\[
L^p(y^kg) = y^{k-2p} \prod_{j=0}^{p-1} (k-2j)[b(k-1-2j+2T^*)+\beta]g.
\]

**Theorem 3.** If the functions \( g_i \) and \( h_i \) are of the form (14), then the polynomial solution of \( L^p(u) = 0 \) for \( p \geq 1 \) is given by
\[ u = \sum_{j=0}^{p-1} (x^{2j}g_j + y^{2j}h_j) \]

where \( 1 > \alpha \in \mathbb{Z} \) and \( 1 > \frac{\beta}{b} \in \mathbb{Z} \).

**Proof.** It is known that \( L^p(x^{2j}g_j) = 0 \) (\( j = 0, 1, \ldots, p-1 \)) from Lemma 3, and \( L^p(y^{2j}h_j) = 0 \) from Lemma 4. Because of the linearity of \( L^p \) the function

\[ u = \sum_{j=0}^{p-1} (x^{2j}g_j + y^{2j}h_j) \]

satisfies the equation \( L^p(u) = 0 \) where \( u \) is a polynomial for \( 1 > \alpha \in \mathbb{Z} \) and \( 1 > \frac{\beta}{b} \in \mathbb{Z} \).

**REFERENCES**


