AN APPLICATION OF THE SUM OF LINEAR OPERATORS IN INFINITE MATRIX THEORY

RABAH LABBAS and BRUNO DE MALAFOSSE

Université Du Havre, I.U.T., B.P 4006, 76610 Le Havre, FRANCE.

(Received Oct. 10, 1997; Accepted Dec. 12, 1997)

ABSTRACT

We present in this work a new application of the sum of linear operators in non differential case. We apply Labbas-Terreni’s [6] approach in infinite-matrix theory and then we give some concrete applications: in the theory of continued fractions, in numerical schemes and in the development of some analytic function on a given basis.

1. INTRODUCTION

In the complex Banach space $E$, the following equation

$$AX + BX - \lambda X = Y, \quad \lambda > 0$$  \hspace{1cm} (1)

where $Y$ is given in $E$ and $A$, $B$ are two closed linear operators with domains $D(A)$, $D(B)$, has been considered by Da Prato and Grisvard [1] and by many authors, under different hypotheses and then many applications in partial differential equation of elliptic, parabolic or hyperbolic type, are given. For example, see R. Labbas and B. Terreni [6] or M. Furhman [3]. Our first aim, in this work, is to present a new application of the sum theory in non differential case. Equation (1) is regarded as an infinite linear system in space $l^\infty$, that is $A + B$ is considered as the sum of two particular infinite matrices defined respectively on $D(A) = s_r$ ($0 < r < 1$) and $D(B) = s_1$ (see section 2) [In infinite matrix theory it means that $A$ belongs to the matrix class $(s_{1/\alpha},s_1)$, see I.J. Maddox [9]]. In our case, we prove that $(-A)$ and $(-B)$ are generators of analytic semigroups not strongly continuous at $t = 0$ since the density of their domains is not true. The classical perturbation theory, as in Kato [4] or in Pazy [10] [for example, the relative boundedness
with respect A or B] is not verified here. So we use the results of Labbas-Terreni [6] in our non commutative situation. The choice of these two infinite matrices is motivated by the resolution of a class of non symmetric linear infinite tridiagonal systems.

The plan of this paper is as follows. In section 2 we recall some results about some regular sets of infinite matrices taken from R. Labbas and B. de Malafosse [5]. Section 3 contains the hypotheses and the main results of the sum-strategy as in Labbas-Terreni [6]. In section 4 we give an example of two infinite matrices A and B regarded as two unbonded linear operators on \( l^\infty = s_1 \); then we study their spectral properties and we show that the Labbas-Terreni [6] sum-strategy can be applied. In section 5 we explicite Lions-Peetre's [8] interpolation spaces between D(A) and E = s_1 for obtaining maximal regularity results for the solution of (1). In the last paragraph we give simple examples which can be regarded by the approach presented here: in the analytic theory of continued fractions, in the study of numerical schemes by finite differences for some differential equations and in the development of some analytic function on a particular basis.

2. THE SPACES \( S_r \) AND \( s_r \)

Consider a positive real number \( r \) and a Banach space \( G \) and denote by \( S_r \) the set of infinite matrices \( M = (a_{nm})_{n,m=1}^{\infty} \) of linear bounded operators from \( G \) to itself, and \( s_r \) the set of infinite column matrices \( X = (x_1, x_2, x_3, ...) = (x_n) \) of elements \( x_n \) in \( G \) which satisfy respectively

\[
\begin{align*}
\sup_{n \geq 1} \left( \sum_{m \geq 1} \|a_{nm}\|_{L(G)} r^{m-n} \right) < \infty \\
\sup_{n \geq 1} \left( \frac{\|x_n\|_G}{r^n} \right) < \infty .
\end{align*}
\]

These spaces are naturally normed by

\[
\begin{align*}
\|M\|_{S_r} &= \sup_{n \geq 1} \left( \sum_{m \geq 1} \|a_{nm}\|_{L(G)} r^{m-n} \right) , \\
\|X\|_{s_r} &= \sup_{n \geq 1} \left( \frac{\|x_n\|_G}{r^n} \right) .
\end{align*}
\]
More generally, we can consider another sequence \( c = (c_n) \), \( c_n > 0 \), instead of \( (r^n) \) in definition of \( S_r \) and \( s_r \); we obtain the spaces \( S_c \) and \( s_c \) characterized by

\[
\begin{align*}
\sup_{n \geq 1} \left( \sum_{m \geq 1} \|a_{mn}\|_{L(G)} \frac{c_m}{c_n} \right) &< \infty , \\
\sup_{n \geq 1} \left( \frac{\|x_n\|_G}{c_n} \right) &< \infty .
\end{align*}
\]

(4)

with the corresponding norms. \( L(G) \) denotes the Banach space of all bounded linear operators from \( G \) into itself. In R. Labbas and B. de Malafosse [5] we have studied some interesting properties of these spaces from which we recall, in particular, the following result:

**Proposition 1.** For any positive real number \( r \) the sets \( s_r \) and \( s_c \) are Banach spaces and the sets \( S_r \) and \( S_c \) are unit Banach algebras.

### 3. SUM OF LINEAR OPERATORS

We simply quote here some results taken from Da Prato-Grisvard [1] and Labbas-Terreni [6]. Consider a complex Banach space \( E \) and two closed linear operators \( A \) and \( B \) defined on their domains \( \text{D}(A) \subset E \) and \( \text{D}(B) \subset E \). Their sum is defined by setting:

\[
SX = AX + BX , \quad X \in D(S) = D(A) \cap D(B)
\]

(5)

Now we shall make the following assumptions on \( A \) and \( B \)

\[
\begin{align*}
&\exists C_A, C_B > 0 , \quad \epsilon_A, \epsilon_B \in ]0,\pi[ \text{ such that } \\
i) \quad \rho(A) \ni \sum_{\epsilon_A} = \{ z \in \mathbb{C} \mid |\text{Arg}(z)| < \pi - \epsilon_A \} \\
&ii) \quad \rho(B) \ni \sum_{\epsilon_B} = \{ z \in \mathbb{C} \mid |\text{Arg}(z)| < \pi - \epsilon_B \} \\
&\quad \| (A - zI)^{-1} \|_{L(E)} \leq C_A / |z| , \quad \forall z \in \sum_{\epsilon_A} - \{0\} , \\
&\quad \| (B - zI)^{-1} \|_{L(E)} \leq C_B / |z| , \quad \forall z \in \sum_{\epsilon_B} - \{0\} , \\
&\quad \epsilon_A + \epsilon_B < \pi .
\end{align*}
\]

(H.1)

If we consider the commutative case:

\[
(A - \xi I)^{-1} (B - \eta I)^{-1} - (B - \eta I)^{-1} (A - \xi I)^{-1}
\]

\[
= \left[ (A - \xi I)^{-1} ; (B - \eta I)^{-1} \right] = 0 ; \quad \forall \xi \in \rho(A) , \quad \forall \eta \in \rho(B)
\]

(6)
and if we assume that \( D(A) \) and \( D(B) \) are densely defined in \( E \), it is well known (see [1]) that the following bounded linear operator (for all \( \lambda > 0 \))

\[
L_\lambda = -\frac{1}{2i\pi} \int_\Gamma (B + zI)^{-1} (A - \lambda I - zI)^{-1} \, dz
\]

coincides with \((S - \lambda I)^{-1}\) where \( S = A + B \). \( \Gamma \) is a simple infinite sectorial curve lying in \( \rho(A - \lambda I) \cap \rho(-B) \). In our applications the commutativity and the density are not true, so we shall assume that:

\[
(H.2) \quad \exists C > 0, \ h \in N, \ (\tau_i)_{i=1}^h, \ (\rho_i)_{i=1}^h \text{ such that } \forall i \geq 1 ; 0 \leq 1 - \tau_i < \rho_i 
\leq 2 \text{ and for } |\lambda|, |\mu| \to \infty ; \lambda \in \rho(A), \mu \in \rho(-B).
\]

Let us put:

\[
\delta = \min_{1 \leq i \leq h} (\tau_i + \rho_i - 1) > 0.
\]

For this commutator see Labbas-Terreni [6], [7]. Now, for any \( \sigma \in ]0,1[ \), let us introduce the real interpolation spaces \( D_A(\sigma,\infty) \) between \( D(A) \) and \( E \) (or \( D_B(\sigma,\infty) \) between \( D(B) \) and \( E \)) characterized in Grisvard [2] by:

\[
D_A(\sigma,\infty) = \left\{ X \in E / \sup_{z \in \Gamma} ||z^\sigma A(A - zI)^{-1} X||_E < \infty \right\},
\]

and which equals, in our case

\[
\left\{ X \in E / \sup_{\theta \in \sigma} ||t^\sigma A(A + tI)^{-1} X||_E < \infty \right\}.
\]

It is a Banach space with the natural following norm:

\[
||X||_{D_A(\sigma,\infty)} = ||X||_E + \sup_{\theta \in \sigma} ||t^\sigma A(A + tI)^{-1} X||_E,
\]

which is equivalent to \( \sup_{\theta \in \sigma} ||t^\sigma A(A + tI)^{-1} X||_E \) when \( A \) is invertible (as in our case). Now we recall the main result proved in [6]:

**Theorem 2.** Under assumptions (H.1) and (H.2) there exists \( \lambda^* \) such that \( \forall \lambda \geq \lambda^* \) and \( \forall Y \in D_A(\sigma,\infty) \) equation \( AX + BX - \lambda X = Y \) has a unique solution \( X \in D(A) \cap D(B) \) which satisfies

i) \( (A - \lambda I)X \in D_A(\theta,\infty) \) \( \forall \theta \leq \min(\sigma,\delta) \),
ii) $BX \in D_A(\theta, \infty) \quad \forall \theta \leq \text{Min}(\sigma, \delta)$,

iii) $(A - \lambda I)X \in D_B(\theta, \infty) \quad \forall \theta \leq \text{Min}(\sigma, \delta)$.

**Remark 1.** Clearly one has a similar result when replacing $A$ by $B$. Notice the unexpected regularity result in the statement iii).

### 4. DEFINITION OF OPERATORS $A$ AND $B$

In this section we consider the Banach space $E = s_1 = l^\infty$ and the two linear operators characterized by the two following infinite matrices:

$$A = \begin{pmatrix}
a & b_1 \\
a^2 & b_2 & O \\
\vdots & \ddots & \ddots \\
0 & \cdots & a^n & b_n & O \\
& & & & \ddots & \ddots \\
& & & & & \ddots & \ddots
\end{pmatrix},$$

such that

\[\begin{cases}
\text{i) } a > 1 \\
\text{ii) } \exists M_A > 0 / \forall n \geq 1 \ |b_n| \leq M_A
\end{cases}\]  \hspace{1cm} (12)

and

$$B = \begin{pmatrix}
\beta_1 \\
\gamma_2 & \beta_2 & O \\
\vdots & \ddots & \ddots \\
0 & \cdots & \gamma_n & \beta_n & O \\
& & & & \ddots & \ddots
\end{pmatrix},$$

under the following assumptions:

\[\begin{cases}
\text{i) } \beta_{2n} = 1 \ , \ \beta_{2n+1} = (2n +1)! \\
\text{ii) } \gamma_{2n} = O(1) \ (n \to \infty) \\
\text{iii) } \gamma_{2n+1} = o(1) \ (n \to \infty)
\end{cases}\]  \hspace{1cm} (14)

**Remark 2.** We have chosen the first diagonal of $A$ in a particular form only to simplify calculus. All the following results in this work are
true if we replace this diagonal by any non decreasing strictly positive sequence \((\alpha_n')\) which tends to infinity. The same remark is true for \(B\); we can replace i) in (14) by

\[
(\beta_n^\alpha) \to L \neq 0 \quad \text{and} \quad \left(\frac{\beta_{2n+1}^\alpha}{\alpha_{2n+1}}\right) \to +\infty.
\]

Then the spectral properties of \(A\) and \(B\) are given by:

**Proposition 3.** In the Banach space \(s_1\) the two linear operators \(A\) and \(B\) are closed and verify

i) \(D(A) = s_{1/\alpha},\)

ii) \(D(B) = s_{1/\beta} = \{X \in s_1 / |x_{2n}| = O(1), |x_{2n+1}| = O\left(\frac{1}{(2n+1)!}\right), (n \to \infty)\},\)

iii) \(D(B) \neq s_1 \) and \(D(A) \neq s_1,\)

iv) There exist positive numbers \(\varepsilon_A, \varepsilon_B, M\) (with \(\varepsilon_A + \varepsilon_B < \pi\)) such that

\[
\| (A - \lambda I) \|_{L(s_1)} \leq \frac{M}{|\lambda|}, \quad \forall \lambda \neq 0 \quad \text{and} \quad |\text{Arg}(\lambda)| \geq \varepsilon_A
\]

\[
\| (B + \mu I) \|_{L(s_1)} \leq \frac{M}{|\mu|}, \quad \forall \mu \neq 0 \quad \text{and} \quad |\text{Arg}(\mu)| \leq \pi - \varepsilon_B.
\]

Statement i) Let \((X_p)\) be a sequence that converges to \(X\) in \(s_1\) and such that \(X_p \in s_{1/\alpha}\) for every \(p\), and \((AX_p)\) converges to \(Y\) in \(s_1\). Let \(X_p = (x_{np})\), then we obviously have

\[
\begin{align*}
\forall n \geq 1 \quad & x_{np} \to x_n \quad (p \to \infty) \\
\forall n \geq 1 \quad & a^n x_{np} + b^n x_{n+1,p} \to y_n \quad (p \to \infty),
\end{align*}
\]

which implies that

\[
\forall n \geq 1 \quad a^n x_n + b^n x_{n+1} = y_n.
\]

However this is not enough to guarantee that \(X = (x_n)\) belongs to \(s_{1/\alpha}\). We deduce this from (12) since \(\exists K > 0\) such that

\[
a^n |x_n| \leq |a^n x_n + b^n x_{n+1} - b^n x_{n+1} | \leq |y_n| + |b^n||x_{n+1}| \leq K.
\]

So the operators \(A\) is closed. The proof is similar for \(B\). It is not difficult to see, from (11), (12), (13) and (14) that \(D(A) = s_{1/\alpha}\) and \(D(B) = s_{1/\beta}\). For the non density of \(D(A)\), it is sufficient to see that the vector \((1, 1, ..., ...) \in s_1\) cannot be approximated by a sequence \(X_n = (x_{np})\) in \(s_{1/\alpha}\) otherwise this would imply that \(x_{n,m}\) tends to 1, when \(m\) tends to
infinity for every \( n \); this is not possible since all sequence in \( s_{1/\alpha} \) tend to zero. An analogous reasoning is true for \( B \). Now it is easy to see that the sequence \( X = (0, 1, 0, 1, ...) \) belongs to \( s_{1/\beta} = D(B) \) but not to \( D(A) \), and the sequence \( X = (1/a^n, 0, 1/a^3, 0, 1/a^5, ...) \) belongs to \( D(A) \) and not to \( D(B) \). This last remark implies that the embedding relative boundedness with respect to \( A \) or \( B \) cannot be applied here.

For fixed \( \varepsilon_0 \in ]0, \pi/2[ \), let us define the infinite sectorial set:

\[
\Pi_{\varepsilon_0} = \{ \lambda \in \mathbb{C} / |\text{Arg}(\lambda)| \leq \varepsilon_0 \} \tag{15}
\]

and for all complex \( \lambda \notin \Pi_{\varepsilon_0} \) let

\[
D_{\lambda} = \left( \frac{1}{a^n - \lambda} \delta_{nm} \right)_{n,m} \quad \delta_{nm} = 1 \text{ if } n = m \text{ and } 0 \text{ elsewhere},
\]

\[
D_{\lambda}(A - \lambda I) = \begin{pmatrix}
1 & q_1 & 0 \\
1 & q_2 & 0 \\
& & \ddots & \ddots \\
& & 0 & 1 & q_n
\end{pmatrix}
\]

with \( q_n = \frac{b_n}{a^n - \lambda} \).

It is easy to see that

\[
\left| \frac{b_n}{a^n - \lambda} \right| \leq \frac{M_\Delta}{a^n \sin \varepsilon_0} \quad \forall \lambda \notin \Pi_{\varepsilon_0},
\]

since

\[
|a^n - \lambda| = d(\lambda, a^n) \geq a^n \sin \varepsilon_0 \quad \forall \lambda \notin \Pi_{\varepsilon_0},
\]

from which we deduce that there exists a larger integer \( n_0 = n(\varepsilon_0) \) such that

\[
n \geq n_0 \Rightarrow \left| \frac{b_n}{a^n - \lambda} \right| \leq \frac{1}{2}.
\]

Now, we consider the following infinite matrix

\[
Q_{\lambda} = \begin{pmatrix}
& & T_{\lambda} & \cdot & \cdot & \cdot & 0 \\
& & & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
& & & & & & \cdot \\
& & & & & & \cdot \\
& & & & & & \cdot \\
O & & & & & & 1
\end{pmatrix},
\]
where $T_\lambda$ is the finite matrix of order $n_0$ defined by

$$T_\lambda = \begin{pmatrix} 1 & q_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & q_{n_0-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -q_1 & -q_1 q_2 & -q_1 q_2 q_3 & \cdots \\ 1 & -q_2 & +q_2 q_3 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -q_{n_0} & \cdots \\ 0 & \cdots & 0 & 1 & -q_{n_0-1} \end{pmatrix}. $$

Then

$$R_\lambda = D_\lambda (A - \lambda I) Q_\lambda = \begin{pmatrix} 1 \\ \vdots \\ 0 \\ 1 & q_{n_0} \\ 1 & q_{n_0+1} \\ \vdots \end{pmatrix}. $$

This product is defined on any space $s_r$, $r > 0$. Hence

$$\|I - R_\lambda\|_{s_1/\alpha} = \sup \left\{ \frac{1}{a} \left| \frac{b_n}{a_n^{\alpha} - \lambda} \right| / n \geq n_0 \right\} \leq \frac{1}{2a} < 1. $$

So that, for any $\lambda \notin \Pi_{\alpha_0}$, $R_\lambda$ is invertible in the Banach algebra $S_{1/\alpha}$ and

$$(A - \lambda I)^{-1} = Q_\lambda R_\lambda^{-1} D_\lambda. \quad (16) $$

Now we can verify that for $Y \in s_1$, we have $D_\lambda Y \in s_{1/\alpha}$, $R_\lambda^{-1}(D_\lambda Y) \in s_{1/\alpha}$, $Q_\lambda R_\lambda^{-1}(D_\lambda Y) \in s_{1/\alpha}$ and

$$\|(A - \lambda I)^{-1} Y\|_{s_1} = \|Q_\lambda R_\lambda^{-1} D_\lambda Y\|_{s_1} \leq \|Q_\lambda\|_{s_1} \|R_\lambda^{-1}\|_{s_1} \|D_\lambda\|_{s_1} \|Y\|_{s_1}, $$

but

$$\|D_\lambda\|_{s_1} = \sup_{n \geq 1} \left( \left| \frac{1}{a_n^{\alpha} - \lambda} \right| \right) \leq \begin{cases} \frac{1}{|\lambda| \sin \theta} & \text{for } \lambda = |\lambda| e^{i\theta} \notin \mathbb{R}^- \\ \frac{1}{|\lambda|} & \text{for } \lambda \in \mathbb{R}^- , \end{cases} $$

since $|a^n - \lambda| = d(\lambda, a^n) \geq d(\lambda, p(\lambda)) = |\lambda| \sin \theta$ for $\lambda = |\lambda| e^{i\theta} \notin \mathbb{R}^-$ where $p(\lambda)$ is the orthogonal projection of $\lambda$ on the real axis.

And

$$\|I - R_\lambda\|_{s_1} = \sup_n \left( \left| \frac{b_n}{a_n^{\alpha} - \lambda} \right| \right) \leq \frac{1}{2}, $$
which implies that
\[ \|R^{-1}_\lambda\|_{s_1} \leq 2. \]

Moreover we can see that there exists some constant \( K \) depending only on \( n_0 \) such that
\[ \|Q\|_{s_1} = \sup \left\{ \left(1 + |q_{n}| + |q_nq_{n+1}| + \cdots + |q_nq_{n+1}\cdots q_{n_0}| \right); \quad n \leq n_0 - 1 \right\} \leq K \]
since all the terms \( q_n \) for \( n \in \{1, 2, \ldots, n_0\} \), are bounded.

So the estimate of \((A - \lambda I)^{-1}\) in statement ii) holds.

For fixed \( \varepsilon_1 \in ]0, \pi/2[ \), let \( \Sigma_{\varepsilon_1} = \{ \mu \in \mathbb{C} / \left| \text{Arg}(\mu) \right| \leq \pi - \varepsilon_1 \} \).
From (14) we deduce that there exists \( n(\varepsilon_1) = n_1 \) such that for any \( \mu \in \Sigma_{\varepsilon_1} \) we have
\[
2k + 1 \geq n_1 \Rightarrow \left| \gamma_{2k+1} \right| \leq \frac{1}{2} \sin \varepsilon_1,
\]
\[
2k \geq n_1 \Rightarrow \left| \frac{\gamma_{2k}}{(1+\mu)(2k-1)!} \right| \leq \frac{1}{2}.
\]

Now consider the following matrices:
\[
D'_\mu = \left( \begin{array}{cccc}
1 & \beta_n + \mu & \delta_{nm} \\
\beta_n + \mu & \ddots & \ddots \\
\delta_{nm} & \ddots & \ddots & \ddots \\
\end{array} \right),
\]
\[
T'_\mu = \left( \begin{array}{cccc}
1 & O & \cdots \\
q_{2} & 1 & \cdots \\
\cdots & \cdots & \ddots & \cdots \\
O & q'_{n_1} & \cdots & 1
\end{array} \right),
\]
\[
Q'_\mu = \left( \begin{array}{cccc}
T'_\mu & \cdots & \cdots & O \\
\cdots & \ddots & \cdots & \ddots \\
\cdots & \cdots & 1 & 1 \\
O & \cdots & \cdots & 1
\end{array} \right),
\]
where \( q'_{n} = \frac{\lambda_n}{\beta_n + \mu} \), (\( n \geq 2 \)).
\[
R'_\mu = Q'_\mu D'_\mu (B + \mu I) = \left( \begin{array}{cccc}
1 & O & \cdots \\
\cdots & \ddots & \cdots & \cdots \\
\cdots & \cdots & 1 & q_{n_1+1} \\
O & q'_{n_1+2} & \cdots & 1
\end{array} \right).
\]

Using (4) one has
\[ \|I - R'_\mu \|_{s_1} = \sup \left\{ \frac{\gamma_n}{\beta_n} \frac{\beta_n}{\beta_n + \mu} ; n \geq n_1 \right\}, \]

\[ = \text{Max} \left( \tau_1, \tau_2 \right) \]

where

\[ \tau_1 = \sup \left\{ \frac{\gamma_{2k}}{1 + \mu} \frac{1}{(2k-1)!} ; k \geq n_1/2 \right\}, \]

\[ \tau_2 = \sup \left\{ \frac{\gamma_{2k+1}}{(2k+1)! + \mu} ; k \geq (n_1 - 1)/2 \right\}. \]

Now it is clear that

\[ \tau_2 \leq 1/2, \]

\[ \tau_1 \leq 1/2, \]

which implies that \( \|I - R'_\mu \|_{s_1} \leq 1/2. \) Thus \( \exists C > 0 \) such that for any \( Y \in s_1 \)

\[ \|(B + \mu)^{-1} Y\|_{s_1} = \|(R'_\mu)^{-1} Q' \mu D' \mu Y\|_{s_1} = \frac{2C}{\mu \sin \varepsilon_1} \|Y\|_{s_1}. \]

The proposition 3 is proved.

The operators (-A) and (-B) verify the hypothesis (H.1) with \( \varepsilon_A = \varepsilon_0 \) and \( \varepsilon_B = \varepsilon_1. \) It is clear that \( \sigma(-A) \) and \( \sigma(B) \) do not intersect, so \( \rho(A) \cup \rho(-B) = C. \)

We have proved that (-A) and (-B) are generators of analytic semigroup \( e^{(-A)t} \) and \( e^{(-B)t} \) not strongly continuous at \( t = 0. \)

Now we shall prove that (H.2) is satisfied under the following additional assumption on A:

\[ \sup_{n \geq 1} \left( \frac{|b_{n-1}|}{a_n} n! \right) < \infty. \]  \hspace{1cm} (17)

**Remark 3.** It follows from (14) that the sequence \( (\gamma_n) \) is bounded by some constant \( C_B, \) so we can always assume that there exists \( \varepsilon_1 \in ]0, \pi/2[ \) such that

\[ \sup_{n \geq 1} \left( |\gamma_{2n}| + \frac{|\gamma_{2n-1}|}{(2n-1)!} \right) = \frac{1}{2} \sin \varepsilon_1, \]

otherwise it would be sufficient to consider \( \sin \varepsilon_1 \frac{1}{2C_B} B \) instead of B.
Proposition 4. Under (12), (14) and (17) there exists a constant $K(e_A e_B) > 0$ such that

$$
\|\mu A(A - \lambda I)^{-1} ; (A^{-1} ; (B + \mu I)^{-1}\|_{L(E)} \leq K(e_A e_B) \sum_{i=1}^{2} \frac{1}{|\lambda|^i |\mu|^\tau_i},
$$

$\forall \lambda \not\in \Pi^A \cap \Pi^B, \forall \mu \not\in \Pi^A \cap \Pi^B$; with $(\tau_1, \rho_1) = (1,1), (\tau_2, \rho_2) = (0,2)$. (The condition $0 \leq 1 - \tau_i < \rho_i \leq 2$ is satisfied).

Proof. Put

$$
C = \mu A(A - \lambda I)^{-1} [A^{-1} ; (B + \mu I)^{-1}] = \mu A(A - \lambda I)^{-1} C_0
$$

and consider the vector-space $S_1^A = \{((B_{nm} A_{nm})/(A_{nm}) \in S_1\}$ which is an unit algebra containing $S_1$, then we can write in this algebra the following equality (for all $\mu \not\in \Pi^A$)

$$
B + \mu I = D_{\mu} + B_0 = D_{\mu}(I + D_{\mu}^{-1}B_0),
$$

where

$$
D_{\mu} = (\beta_n + \mu) \delta_{nm}, B_0 = \begin{pmatrix} 0 & 0 \\ \gamma_2 & 0 \\ \gamma_3 & 0 \\ O & . \end{pmatrix},
$$

since $(I + D_{\mu}^{-1}B_0) \in S_1$. It follows from (14), and (17) that

$$
\|D_{\mu}^{-1} B_0\|_{S_1} = \sup_{n \geq 2} \left( \frac{|\gamma_n|}{|\beta_n + \mu|} \right) \leq \frac{1}{2},
$$

which gives the classical Neumann's series (in the Banach algebra $S_1$):

$$
(B + \mu I)^{-1} = \sum_{n=0}^{\infty} (-1)^n (D_{\mu}^{-1} B_0)^n (D_{\mu}^{-1}),
$$

from which we have:

$$
C_0 = [A^{-1} ; (B + \mu I)^{-1}] = A^{-1} (B + \mu I)^{-1} - (B + \mu I)^{-1} A^{-1}
$$

(18)

$$
= D_{\mu}^{-1}A^{-1} - A^{-1}D_{\mu}^{-1} - [D_{\mu}^{-1}B_0 D_{\mu}^{-1}] A^{-1} + A^{-1} [D_{\mu}^{-1}B_0 D_{\mu}^{-1}]
$$

$$
+ \sum_{n=2}^{n-2} (-1)^n [D_{\mu}^{-1}B_0^2] (D_{\mu}^{-1}B_0 D_{\mu}^{-1} A^{-1} - A^{-1} (D_{\mu}^{-1}B_0) (D_{\mu}^{-1}B_0 D_{\mu}^{-1})]^{n-2},
$$

$$
= C_1 + R,
$$
where $C_1$ is the sum of the first four elements of $C_0$:

$$
C_1 = D^{-1}_\mu (I - B_0 D^{-1}_\mu) A^{-1} - A^{-1} D^{-1}_\mu (I - B_0 D^{-1}_\mu)
= K_\mu A^{-1} - A^{-1} K_\mu
= (c_{nm})_{n,m=1,...}
$$

with

$$
K_\mu = D^{-1}_\mu (I - B_0 D^{-1}_\mu) = \begin{pmatrix}
\zeta_1 & 0 \\
\eta_2 & \zeta_2 \\
\eta_3 & 0
\end{pmatrix}
$$

and

$$
\begin{aligned}
\zeta_n &= \frac{1}{\beta_n + \mu} \\
\eta_n &= -\frac{\gamma_n}{(\beta_{n-1} + \mu)(\beta_n + \mu)}.
\end{aligned}
$$

Now, using the explicite following formula for the upper triangular infinite matrix

$$
A^{-1} = (\alpha_{nm})
$$

with

$$
\alpha_{nm} = (-1)^{m-n}(b_n b_n \ldots b_{m-1})a \frac{a^{-(m-n)(m+1)}}{2}, \forall m > n \geq 1
$$

$$
\alpha_{nm} = \frac{1}{a^n},
$$

it follows that

$$
\begin{aligned}
c_{n,n-1} &= (\alpha_{n-1,n-1} - \alpha_{n})\eta_n \quad n > 1 \\
c_{nm} &= \alpha_{nm}(\zeta_n - \zeta_m) + \eta_n \alpha_{n-1,m} - \alpha_{n,m+1}\eta_{m+1} \quad m \geq n
\end{aligned}
$$

which yields

$$
\begin{aligned}
c_{n,n-1} &= \frac{\gamma_n}{(\beta_{n-1} + \mu)(\beta_n + \mu)} \frac{a^{n-1}}{a} \quad \forall n > 1, \\
c_{nm} &= \alpha_{nm} \left[ \frac{\beta_m - \beta_n}{(\beta_n + \mu)(\beta_m + \mu)} \right] \\
&\quad + \alpha_{nm} \left[ \frac{b_m}{a^{m+1}} (\beta_m + \mu)(\beta_{m+1} + \mu) - \frac{b_{m-1}}{a^{n-1}} (\beta_{n-1} + \mu)(\beta_n + \mu) \right]
\end{aligned}
$$
\[ = \alpha_{nm} k_{nm} + \alpha_{nm}^' k_{nm}^' \quad \forall m \geq n. \]

We easily verify that

\[ |k_{nm}^'| = \left| \frac{b_m}{a^{m+1}} \frac{\gamma_{m+1}}{(\beta_m + \mu)(\beta_{m+1} + \mu)} - \frac{b_{m-1}}{a^{m-1}} \frac{\gamma_n}{(\beta_{m-1} + \mu)(\beta_n + \mu)} \right| \leq \frac{M_A M_B}{|\mu|^2 \sin^2 \varepsilon_i} \]

So the commutator \( C_1 \) in (18) may be written as:

\[ C_1 = C_{11} + C_{12} + C_{13}, \]

with respectively

\[ C_{11} = \begin{pmatrix} 0 & 0 & 0 \\ c_{21} & 0 & 0 \\ 0 & c_{32} & 0 \end{pmatrix}; \quad C_{12} = (\alpha_{nm} k_{nm}); \quad C_{13} = (\alpha_{nm} k_{nm}^') \quad \forall m \geq n. \]

Using a direct calculation and the fact that in infinite matrix theory, the property \((M)^{-1} = (M^{-1})^t\) holds for any invertible matrix \(M\), the matrix \(A(A - \lambda I)^{-1}\) can be written as a sum of a diagonal matrix and an upper triangular matrix

\[ A(A - \lambda I)^{-1} = D_1 + T_1, \]

where

\[ D_1 = \begin{pmatrix} a_n^{-1} \delta_{nm} \\ a_n^{-1} - \lambda \end{pmatrix}, \quad T_1 = (t_{nm}), \]

and

\[ t_{nm} = \lambda \frac{(-1)^{m-n}(b_n b_{n+1} \cdots b_{m-1})}{(a_n^{-1} - \lambda)(a_{n+1}^{-1} - \lambda) \cdots (a_{m-1}^{-1} - \lambda)(a_m^{-1} - \lambda)} \quad \forall m \geq n + 1. \]

Now we have in algebra \( S_1 \):

\[ \| (D_1 + T_1)(C_{11} + C_{12} + C_{13}) \| \leq \| D_1 C_{11} \| + \| D_1 C_{12} \| + \| D_1 C_{13} \| + \| T_1 C_{11} \| + \| T_1 C_{12} \| + \| T_1 C_{13} \| \]

It is easily seen, from hypotheses (12) and (17) that there exists a constant \( K_0 \) such that
∀m ≥ n ⇒ [b_n b_{n+1}...b_m] ≤ K_0,

(20)

then for any λ ∈ Π_{b_a} and any μ ∈ Π_{b_a}, we have

\[ \|D_1C_{11}\|_S_1 = \sup_{n \geq 2} \frac{a^n}{a^n - \lambda} c_{n,n-1} = \sup_{n \geq 2} \frac{a^n}{a^n - \lambda} (\beta_{n-1} + \mu)(\beta_n + \mu) \left(\frac{a}{a^n - \lambda}\right) \left(\frac{a}{a^n - \lambda}\right) \leq \frac{K_1(e_0,e_1)}{|\lambda| |\mu|^2}, \]

\[ \|D_1C_{12}\|_S_1 = \sup_{n \geq 2} \sum_{m=n+1}^{\infty} \frac{(b_{n+1}b_{n+1}...b_m)}{a_n a_{n+1}...a_m} \frac{1}{a^n - \lambda} \left[ \frac{\beta_m - \beta_n}{(\beta_n + \mu)(\beta_m + \mu)} \right] \leq \frac{K_0K_2(e_0,e_1)}{|\lambda| |\mu|^2} \sup_{n \geq 2} \sum_{m=n+1}^{\infty} \frac{|b_{m-1}m|}{a_{n+1}...a_{m-1}m}, \]

and from (17), there exists some constant K' such that

\[ \|D_1C_{12}\|_S_1 \leq K' \frac{K_0K_2(e_0,e_1)}{|\lambda| |\mu|^2} \sum_{m=n+1}^{\infty} \frac{1}{a_{n+1}...a_m} \leq K' \frac{K_0K_2(e_0,e_1)}{|\lambda| |\mu|^2} \sum_{m=n+1}^{\infty} \frac{1}{a_{n+m}} \leq K' \frac{K_0K_2(e_0,e_1)}{|\lambda| |\mu|^2} \frac{a}{a - 1}. \]

On the other hand, it follows from (19)

\[ \|D_1C_{13}\|_S_1 = \sup_{n \geq 2} \sum_{m=n+1}^{\infty} \ \left| a_{n \text{,} m} \frac{a^n}{(a^n - \lambda)} \right| \leq \frac{K_0M_A M_B}{|\lambda|^2 |\mu|^2} \left( \sup_{n \geq 2} \left| \frac{b_n}{(a^n - \lambda)} \right| + \sum_{m=n+2}^{\infty} \left| \frac{b_{n+1}b_{n+1}...b_m}{a_{n+1}...a_m} \frac{1}{(a^n - \lambda)} \right| \right) \leq \frac{K_3(e_0,e_1)}{|\lambda| |\mu|^2} \left( \sup_{n \geq 2} \sum_{m=n+1}^{\infty} \frac{1}{a_{n+1}...a_m} \right) + K'' \leq \frac{K_3(e_0,e_1)}{|\lambda| |\mu|^2} \left( \frac{a}{a - 1} + K'' \right). \]

Finally
\[ \| T_{1}C_{11} \|_{S_{1}} \leq \| T_{1} \|_{S_{1}} \| C_{11} \|_{S_{1}} \]

\[ = \sup_{n \geq 2} |c_{n,n-1}| \sup_{n \geq 2} \left( \sum_{m=n+1}^{\infty} \frac{\lambda (a^{n+1} - \lambda) \ldots (a^{m-1} - \lambda)}{(a^{n} - \lambda)(a^{n+1} - \lambda)} \right) \frac{1}{(a^{m} - \lambda)} \]

\[ \leq \sup_{n \geq 2} |c_{n,n-1}| \sup_{n \geq 2} \left( \frac{\lambda b_{n+1} \ldots b_{m-1}}{(a^{n} - \lambda)(a^{n+1} - \lambda)} \right) \frac{|b_{n+2} \ldots b_{m-1}|}{|a^{n} - \lambda| |a^{n+2} - \lambda| \ldots |a^{m} - \lambda|} \]

\[ \leq \frac{\lambda |M_{A}M_{B}|}{\lambda |M_{B}K_{0}|} \sum_{n \geq 2} \frac{1}{\lambda |M_{B}| \mu |M_{A}| \sin \epsilon_{1} \sin \epsilon_{0}} \frac{1}{\sin \epsilon_{1}} \frac{1}{\sin \epsilon_{0}} \sum_{m=n+2}^{\infty} \frac{1}{(m+n+2) \sin \epsilon_{0}} \]

\[ \leq K_{4} \left( \frac{\epsilon_{A} \epsilon_{B}}{\lambda |M_{B}| \mu |M_{A}| \sin \epsilon_{1} \sin \epsilon_{0}} \frac{1}{\sin \epsilon_{1}} \frac{1}{\sin \epsilon_{0}} \right) \]

To bound the next step we shall need the following formula

\[ \| T_{1}C_{12} \|_{S_{1}} \leq \sup_{n \geq 2} \left( \sum_{m=n+2}^{\infty} \left( \sum_{j=m+1}^{\infty} \frac{\lambda b_{n+1} \ldots b_{j-1}}{(a^{n} - \lambda)(a^{j} - \lambda)} b_{j} \ldots b_{m-1} \frac{(\beta_{j} - \beta_{m})}{(\beta_{m} + \mu)(\beta_{m} + \mu)} \right) \right) ; \]

we use also the fact that \((\beta_{j} - \beta_{m}) \leq m! \) and (17). Therefore it follows

\[ \| T_{1}C_{12} \|_{S_{1}} \leq \frac{K_{0}C}{|\lambda| |\mu|^{2} \sin \epsilon_{B}} \sum_{n \geq 2} \sum_{m=n+1}^{\infty} \frac{1}{a^{m+n+2} \sin \epsilon_{0}} \frac{1}{\sin \epsilon_{1}} \frac{1}{\sin \epsilon_{0}} \]

\[ \leq K_{5} \left( \frac{\epsilon_{A} \epsilon_{B}}{|\lambda| |\mu|^{2} \sin \epsilon_{B} \sin \epsilon_{1} \sin \epsilon_{0}} \right) \]

The last term is easily bounded as

\[ \| T_{1}C_{13} \|_{S_{1}} \leq \| T_{1} \|_{S_{1}} \| C_{13} \|_{S_{1}} \leq K_{6} \left( \frac{\epsilon_{A} \epsilon_{B}}{|\lambda| |\mu|^{2}} \right) \]

Now if we group all these estimates, we deduce

\[ \| \mu A (A - \lambda I)^{-1} [A^{-1}; (B + \mu I)^{-1}] \| \]

\[ = \| \mu A (A - \lambda I)^{-1} (C_{0}) \| \]

\[ \leq \| \mu A (A - \lambda I)^{-1} (C_{1} + R) \| \]

\[ \leq \| \mu A (A - \lambda I)^{-1} C_{1} \| + \| \mu A (A - \lambda I)^{-1} R \| \]

\[ \leq \| \mu (D_{1} + T_{1}) (C_{12} + C_{13}) \| + \| \mu A (A - \lambda I)^{-1} R \| \]

\[ \leq K_{5} \left( \frac{\epsilon_{A} \epsilon_{B}}{|\lambda| |\mu|^{2} \sin \epsilon_{B} \sin \epsilon_{1} \sin \epsilon_{0}} \right) \]

\[ \leq K \left( \frac{1}{|\lambda| |\mu|^{2}} \right). \]
We have used the fact that \( \| A(A - \lambda I)^{-1} R \| = O(\| R \|) = O\left( \frac{1}{|\mu|} \right) \) when \( |\lambda|, |\mu| \) tend to infinity, which can be deduced from (18). The proposition 4 is proved and the numer \( \delta \) defined in (9) is equal to 1.

5. REAL INTERPOLATION SPACES

We shall apply theorem 2 given in section 3, so we must explicite the interpolation Banach space \( D_A(\theta, \infty) \) between \( D_A = s_{1/\alpha} \) and \( E = s_1 \). We shall use property (10).

**Proposition 5.** For any \( \theta \in ]0,1[ \) the space \( D_A(\theta, \infty) \) coincides with \( s_{1/\alpha} \).

**Proof.** We shall prove only that \( s_{1/\alpha} \subset D_A(\theta, \infty) \); for the other embedding, the same techniques can be used. Given \( X = (x_{m}) \in s_{1/\alpha} \), then

\[
\forall m \geq 1 \quad |x_m| \leq \| X \|_{s_{1/\alpha}} \left( \frac{1}{|\theta|} \right)^m
\]

(21)

Since \( A \) is invertible, we must only estimate the semi-norm \( \sup_{t \to 0} \| t A(A + tl)^{-1} X \|_E \).

Let \( t_0 \) be such that

\[
\forall t \geq t_0 \Rightarrow M_A \leq \frac{1}{2}
\]

(22)

then it suffices to estimate \( \sup_{t \to 0} \| t A(A + tl)^{-1} X \|_E \). Let us write that \( A(A + tl)^{-1} = I - t(A + tl)^{-1} = (r_{nm}^t)_{n,m=1,...} \) where

\[
r_{nm}^t = \begin{cases} 
0 & \text{if } m < n \\
\frac{a^n}{a^n + t} & \text{if } m = n \\
\left(-1\right)^{m-n-1} \frac{b_n b_{n+1} \cdots b_{m-1}}{(a^n + t)(a^{n+1} + t) \cdots (a^m + t)} & \text{if } m > n
\end{cases}
\]

Then from \( t^\theta A(A + tl)^{-1} X = Y = (y_n^t)_{n \geq 1} \), we get

\[
y_n^t = \sum_{m \geq n} r_{nm}^t x_m.
\]
(The convergence is justified by (16) for example). Recalling (21) and (22) it yields
\[
\begin{align*}
Y_n & \leq t^{\theta} \sum_{m=2n}^{\infty} |r_m^l x_m| \\
& \leq \frac{t^{\theta} a_n^{\theta}}{a_n + t} |x_n| + \sum_{m=2n+1}^{\infty} \frac{t |b_n b_{n+1} \ldots b_{m-1} | t^{\theta} |x_m|}{(a_n + t)(a_{n+1} + t) \ldots (a_m + t)} \\
& \leq \frac{t^{\theta} a_n^{\theta}}{a_n + t} \frac{||X||_{S(1/\theta)}}{a_n^{\theta}} + \sum_{m=2n+1}^{\infty} \frac{t |b_n b_{n+1} \ldots b_{m-1} | t^{\theta} ||X||_{S(1/\theta)}}{(a_n + t)(a_{n+1} + t) \ldots (a_m + t)a_m^{\theta}}
\end{align*}
\]

then, from
\[
\sup_{\theta > 0} \frac{t^{\theta} a_n^{\theta}}{a_n + t} = (1 - \theta)^{1-\theta} \frac{\theta}{\theta a_n^{\theta}}
\]

we have
\[
|y_n| \leq K_1 ||X||_{S(1/\theta)} + K_2 \sum_{m=2n+1}^{\infty} \frac{(M_{\lambda})^{m-n}}{m_n^{\theta}} ||X||_{S(1/\theta)} \\
\leq K_1 ||X||_{S(1/\theta)} + (M_{\lambda})^{-1} K_2 \sum_{j=0}^{\infty} \frac{1}{(2a)^j} ||X||_{S(1/\theta)}.
\]

Applying theorem 2, one has

**Theorem 6.** Let A and B be the two infinite matrices defined by (1), (13) such that (12), (14) and (17) hold. Then there exists \( \lambda^* \) such that \( \forall \lambda \geq \lambda^* \) and \( \forall Y \in s_{\alpha} \) the linear infinite system \([-A - B - \lambda I].X = Y \) has a unique solution \( X \in s_{1/\alpha} \cap s_{1/\beta} \) such that:

i) \((A + \lambda I)X \in s_{1/\alpha} \cap s_{1/\beta} \),

ii) \(BX \in s_{1/\alpha} \cap s_{1/\beta} \).

In our case it is easily seen that \( s_{1/\alpha} \cap s_{1/\beta} \) coincides with \( s_c \), where the sequence \((c_n)\) is defined by

\[
c_n = \begin{cases} 
\frac{1}{2k} & \text{if } n = 2k \\
\frac{1}{(2k + 1)!} & \text{otherwise}
\end{cases}
\]
6. EXAMPLES AND REMARKS

6.1. In continued fractions theory. Let us define the following particular sequences

\[ \omega_{n-1} = b_{n-1} = \gamma_n = \frac{c_n a^n}{n!}, \quad n \geq 2 \]

\[ \kappa_n = \begin{cases} 
2k + 1 & \text{if } n = 2k \\
2k+1! + (2k + 1)! & \text{if } n = 2k + 1,
\end{cases} \]

where the sequence \((c_n)\) is assumed to be bounded. Consider now the following infinite system:

\[
\begin{align*}
(\kappa_1 + \lambda)x_1 - \omega_1 x_2 &= y_1 \\
- \omega_1 x_1 + (\kappa_2 + \lambda)x_2 - \omega_2 x_3 &= y_2 \\
- \omega_{m-1} x_{m-1} + (\kappa_m + \lambda)x_m - \omega_m x_{m+1} &= y_m \\
\cdots &= \cdots
\end{align*}
\]

For the particular second member \((1, 0, 0, \ldots) \in s_{1/\alpha}\), this system gives formally the following continued fraction

\[ x_1 = \frac{1}{\kappa_1 + \lambda - \frac{\omega_1^2}{\kappa_2 + \lambda - \frac{\omega_2^2}{\cdots}}} \]

and in virtue of theorem 6, we see that this fraction converges, since, by using the right inverse of \(A + B + \lambda I\), the first element \(x_1\) of the solution \(X = (x_1, x_2, \ldots)\) coincides exactly with the first component of

\[(A + B + \lambda I)^{-1}(1, 0, 0, \ldots),\]

with the following regularity:

\[ \forall \theta \in ]0, 1[ \sup_{n \geq 1} \left( a^n \left| a^n x_n + b^n x_{n+1} \right| \right) < \infty. \]

6.2. In numerical schemes. Our approach can be applied in studying differential equations that are approximated by finite difference schemes of type

\[ (*) \begin{cases} 
a_{n-1} u_{n-1} + (b_n + \lambda)u + c_{n+1} u_{n+1} = f_n \\
n = 0, \pm 1, \pm 2, \pm 3, \ldots
\end{cases} \]
where \((f_n)\) is given in space \(l^\theta\). The hypotheses concerning sequences \((a_n), (b_n), (c_n)\) mean that \((*)\) is of elliptic type. Theorem 6 gives the resolution of \((*)\) in the adapted space \(l^\theta\), see [5].

6.3. Development of some analytic function on a some given basis. For \(\theta \in ]0,1[, \alpha > 1\) and \(\rho \in ]0,1[\) consider the analytic function

\[
f(z) = \frac{a^\theta}{a^\theta - z} = \sum_{n=0}^{\infty} \frac{1}{a^{n \theta}} z^n \quad \text{for } |z| \leq \rho
\]

then thanks to theorem 6, there exists a unique sequence \(X = (x_n)_{n \geq 0}\) belonging to the space \(s_c\) (defined in (23)) such that

\[
f(z) = \frac{a^\theta}{a^\theta - z} = \sum_{n=0}^{\infty} \frac{1}{a^{n \theta}} z^n = x_0 + \sum_{n=1}^{\infty} x_n \left[ b_{n-1} z^{n-1} + (a^n + \beta_n + \lambda)z^n + \gamma_{n+1} z^{n+1} \right]
\]

for \(|z| \leq \rho\) and a large \(\lambda > 0\). Applying the result of regularity, we have also:

\[
\sup_{n \geq 0} (a^{n \theta} a_n x_n + b_n x_{n+1}) < \infty.
\]

In fact, we can see that equation (24) is equivalent to (1) with \(Y = (1, 1/\alpha^\theta, 1/\alpha^{2 \theta}, \ldots)\) and \(b_0 = 0; (b_n)_{n \geq 1}, (\beta_n)_{n \geq 1}, (\gamma_n)_{n \geq 2}\) are three sequences verifying (12), (14) and (17). This justified by the absolute convergence of double series \(\sum_{n \geq 1} \sum_{m \geq 1} |a_{nm}(\lambda)| |x_m| |z| \) when \(|z| \leq \rho\) and \((x_n) \in s_c\) (see(23)), here \(a_{nm}(\lambda)_{nm} = -A - B - \lambda I\) and \(A, B\) are defined as in (11), (13).

Remark. When \(E\) is a Hilbert space, M. Fuhrman [3] has proved, under (H.1) and (H.2), that for large \(\lambda\) and for any \(Y \in E\) the unique solution \(X \in D(A) \cap D(B)\) of (1), exists and satisfies

\[
\|A.X\|_E + \|B.X\|_E \leq K(\lambda)\|Y\|_E,
\]

this estimate is important for studying stability and convergence of numerical schemes of type (\(*\)). This approach, in Hilbert case, gives also the resolution of infinite linear systems when the second member is not regular but only in \(E\).
REFERENCES


