ON THE SUBRINGS OF THE RING OF ANALYTIC FUNCTIONS AND CONFORMALLY EQUIVALENCE

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ABSTRACT

We consider the discrete sets $D_i \subseteq G_i$ of the regions $G_i$ in the complex plane $\mathcal{E}$ and study the subrings of the rings of analytic functions $A(G_i)$ corresponding to these discrete sets $D_i$ ($i=1,2$). Furthermore, we prove that the sets of zeros of the functions which map onto each other under the $\mathcal{E}$-isomorphism $\Phi: A(G_1) \rightarrow A(G_2)$, are also mapped onto each other by a conformal mapping $\varphi: G_2 \rightarrow G_1$, where $\Phi(f) = f \circ \varphi$.

1. INTRODUCTION

Helmer had shown that finitely generated ideal in the ring of entire functions was a principle ideal [2]. After this work, some other authors have started to work on the rings of analytic functions, and conformal equivalence was characterized algebraically.

Let $G_1$ and $G_2$ be two domains in the complex plane, and let $A(G_1)$ and $A(G_2)$ be the rings of analytic functions of $G_1$ and $G_2$ respectively. If these exists a $\mathcal{E}$-isomorphism between $A(G_1)$ and $A(G_2)$, then $G_1$ and $G_2$ are conformally equivalent [1]. The problem was generalized to open Riemann surfaces $G_1$ and $G_2$ [4]. It was shown that two domains $G_1$ and $G_2$ in the complex plane were conformally equivalent if the rings $B(G_1)$ and $B(G_2)$ of all bounded analytic functions defined on them were algebraically $\mathcal{E}$-isomorphic [3]. When we discuss the rings $B(G_i)$ ($i=1,2$), it is always assumed that $G_i$ is bounded and has the following property: for any $z \in \partial G_i$, boundary of $G_i$, there exists a function $f \in B(G_i)$ for which $z$ is an unremovable singularity. It is proved that if there is a
$\mathcal{C}$-isomorphism between $A(G_1)$ and $A(G_2)$, then the sets $G_1$ and $G_2$ are conformally equivalence [5].

**Definition 1.1.** Let $S$ be any non-empty subset of the complex plane $\mathbb{C}$ and $f: S \to \mathbb{C}$ be a function. If $f$ is an analytic function in the domain which contains $S$, $f$ is called an analytic function is $S$.

Let $G$ be any non-empty subset of $\mathbb{C}$ and $A(G)$ be the set of single-valued analytic functions on $G$. The set $A(G)$ is a ring with respect to two binary operations which are defined by $(f+g)(z) = f(z)+g(z)$ and $(fg)(z) = f(z)g(z)$.

**Definition 1.2.** Let $G_1$ and $G_2$ be two non-empty subsets of $\mathbb{C}$. If the mapping $\varphi: G_1 \to G_2$ is analytic and bijective, then $\varphi$ is called a conformal mapping from $G_1$ to $G_2$. In this case, $G_1$ and $G_2$ are called conformally equivalent.

In this work, an analytic function means that it is a single-valued analytic function.

Let $a \in G$ be an arbitrary but fixed point. The set $M_a = \{f \in A(G): f(a) = 0\}$ is a maximal ideal which is generated by $z-a$ from $A(G)$. $M_a$ is called a fixed maximal ideal of $A(G)$, and all other maximal ideals of $A(G)$ are called free maximal ideal.

**Definition 1.3.** Let $G$ be a region (or a set) in the complex plane and $D \subset G$. If $D$ has no limit point in $G$, then $D$ is called a discrete subset of $G$.

**Theorem 1.4.** Let $G_1$ and $G_2$ be two subsets of $\mathbb{C}$, and $\Phi$ be an $\mathcal{C}$-isomorphism from $A(G_1)$ onto $A(G_2)$. Then $\Phi$ induces a mapping $\varphi: G_2 \to G_1$, defined by $\Phi(f) = f \circ \varphi$, and $\varphi$ is a conformal mapping of $G_2$ onto $G_1$ [5].

**Theorem 1.5.** Let $G_1$ and $G_2$ be two subsets of $\mathbb{C}$, and $\varphi: G_2 \to G_1$ be a conformal mapping. Then the mapping $\Phi$ defined by $\Phi(f) = f \circ \varphi$ is a $\mathcal{C}$-isomorphism from $A(G_1)$ onto $A(G_2)$ [5].
2. SUBRINGS OF THE RING A(G)

In this section, subrings of the ring A(G) corresponding to the discrete sets will be investigated.

**Theorem 2.1.** Let

\[ A_{D_1}(G_1) = \{ f \in A(G_1): f(u) = \text{constant, for all } u \in D_1 \} \]

and

\[ A_{D_2}(G_2) = \{ g \in A(G_2): g(v) = \text{constant, for all } v \in D_2 \} \]

where \( D_1 \) and \( D_2 \) are discrete subsets of \( G_1 \) and \( G_2 \), respectively. Let \( \varphi: G_2 \rightarrow G_1 \) be a bijective analytic mapping. If \( \varphi(D_2) = D_1 \), then \( \Phi: A_{D_1}(G_1) \rightarrow A_{D_2}(G_2) \), \( \Phi(f) = f \circ \varphi \) is a \( \mathcal{C} \)-isomorphism.

**Proof:** It is easily seen that \( \Phi(f_1 f_2) = \Phi(f_1) \Phi(f_2) \) and \( \Phi(f_1 + f_2) = \Phi(f_1) + \Phi(f_2) \). Hence \( \Phi \) is a homomorphism. Since \( g \) is an element of \( A_{D_2}(G_2) \), \( g \circ \varphi^{-1} \in A_{D_1}(G_1) \). If \( \varphi^{-1}(c) \in D_2 \), \( c \in D_1 \), then \( (g \circ \varphi^{-1})(c) = g(\varphi^{-1}(c)) \) = constant. Therefore, for all \( g \in A_{D_2}(G_2) \), there exists \( g \circ \varphi^{-1} \in A_{D_1}(G_1) \) such that \( \Phi(g \circ \varphi^{-1}) = (g \circ \varphi^{-1}) \circ \varphi = g \). Hence \( \Phi \) is onto. On the other hand, since

\[ \Phi(f_1) = \Phi(f_2) \Rightarrow f_1 \circ \varphi = f_2 \circ \varphi \Rightarrow f_1 = f_2 \]

\( \Phi \) is one to one. It is obvious that the constants are invariant under the isomorphism \( \Phi \). Hence \( \Phi \) is a \( \mathcal{C} \)-isomorphism.

The following theorem gives us a relation between the ring \( A(G) \) and the subring \( A_D(G) \).

**Theorem 2.2.** If a discrete set \( D \subset G \) contains only one element, i.e., \( D = \{ a \} \), then \( A_D(G) = A(G) \).

**Proof:** From definition of \( A_D(G) \), we have \( A_D(G) \subset A(G) \). On the other hand, suppose that \( f \in A(G) \). In this case, since \( f(a) = \text{constant} \), \( f \) is an element of \( A_D(G) \). Then \( A(G) \subset A_D(G) \) holds. Therefore, \( A(G) = A_D(G) \).

By this theorem, we can say that we will be able to work on the subring \( A_D(G) \) instead of the ring \( A(G) \). If \( D \) has only one element, we can take the set \( A_D(G) \) for \( A(G) \).
Corollary 2.3. Let $D_1$ and $D_2$ be two discrete subsets of $G$. If $D_1 \subset D_2$

$$A_{D_1}(G) = \{ f \in A(G): f(u) = \text{constant, for all } u \in D_1 \}$$

and

$$A_{D_2}(G) = \{ g \in A(G): g(v) = \text{constant, for all } v \in D_2 \}$$

then $A_{D_2}(G) \subset A_{D_1}(G)$.

**Proof:** Let $f \in A_{D_2}(G)$. Then for all $v \in D_2$, $f(v) = \text{constant}$. Since $D_1 \subset D_2$ and $f(u) = \text{constant}$ for all $u \in D_1$, $f \in A_{D_1}(G)$. Hence, we have $A_{D_2}(G) \subset A_{D_1}(G)$.

Let $D_f \subset G$ be a set of zeros of $f \in A(G)$. Now we can give the following theorem on maximal ideals of $A_{D_f}(G)$.

**Theorem 2.4.** Let $f \in A(G)$ and $D_f$ be a finite set. Furthermore suppose that

$$A_{D_f}(G) = \{ g \in A(G): g(u) = \text{constant, for all } u \in D_f \}$$

Then

$$\{ g(z) \prod_{u \in D_f} (z-u): g \in A(G) \}$$

is a maximal ideal of $A_{D_f}(G)$.

**Proof.** It is calisy shown that $J = \{ g(z) \prod_{u \in D_f} (z-u): g \in A(G) \}$ is an ideal of $A_{D_f}(G)$. Now, we will show that this ideal is maximal. Let us consider the mapping $\psi_u: A_{D_f}(G) \to \mathcal{E}$, $\psi_u(g) = g(u)$, $u \in D_f$. It is clear that the mapping $\psi_u$ is a homomorphism. There exists $h = f^0 + ce A_{D_f}(G)$ such that $\psi_u(h) = h(u) = e$, $(c \in \mathcal{E})$, hence $\psi_u$ is onto. At the same time

$$\text{Ker} \psi_u = \{ h \in A_{D_f}(G): \psi_u(h) = h(u) = 0 \}.$$ 

Since $h(u) = 0$ for all $u \in D_f$ and $h \in A_{D_f}(G)$, we get $h(z) = g(z) \prod_{u \in D_f} (z-u)$, where $g \in A(G)$. Hence $\text{Ker} \psi_u = J$. According to the first isomorphism theorem

$$A_{D_f}(G)/J \cong \mathcal{E}.$$
Hence, $J$ is a maximal ideal.

$M_a = \{g \in A(G) : g(a) = 0\}$ is a maximal ideal of $A(G)$, where $a \in G$. In Theorem 2.4, taking a discrete set instead of $a$, this result is generalized for maximal ideals.

**Theorem 2.5.** Let $D_{f_1}$ and $D_{f_2}$ be sets of zeros of $f_1 \in A(G_1)$ and $f_2 \in A(G_2)$, respectively. Moreover, suppose that the mapping $\Phi : A(G_1) \to A(G_2)$ defined by $\Phi(f) = f \circ \varphi$ is a $C$-isomorphism. If $\Phi(f_1) = f_2$, then $\varphi(D_{f_2}) = D_{f_1}$.

**Proof.** From Theorem 2.4, $G_2$ and $G_1$ are conformally equivalent, i.e., there exists a mapping $\varphi : G_2 \to G_1$ which is analytic and bijective. From the hypothesis $f_2 = f_1 \circ \varphi$. If $a \in D_{f_2}$, then

$$0 = f_2(a) = (f_1 \circ \varphi)(a) = f_1(\varphi(a)).$$

Thus, $\varphi(a) \in D_{f_1}$. Since $a$ is an arbitrary element of $D_{f_2}$, we have that $\varphi(D_{f_2}) \subseteq D_{f_1}$. On the other hand, if $d \in D_{f_1}$, then there exists $c \in G_2$ such that $\varphi(c) = d$. Hence

$$0 = f_1(d) = f_1(\varphi(c)) = (f_1 \circ \varphi)(c) = f_2(c)$$

and $c \in D_{f_2}$. Thus $D_{f_1} \subseteq \varphi(D_{f_2})$. Then the result follows.

We can give the following theorem as a corollary of Theorem 2.1 and 2.5.

**Theorem 2.6.** If $F : A(G_1) \to A(G_2)$ is a $C$-isomorphism and $\Phi(f_1) = f_2$, then

$$A_{D_{f_1}}(G_1) \cong A_{D_{f_2}}(G_2).$$

**Proof.** From the hypothesis and Theorem 2.4, $G_2$ and $G_1$ are conformally equivalent, i.e., there exists a mapping $\varphi : G_2 \to G_1$ which is analytic and bijective. According to Theorem 2.5, $\varphi(D_{f_2}) = D_{f_1}$. From Theorem 2.1, we have

$$A_{D_{f_1}}(G_1) \cong A_{D_{f_2}}(G_2).$$
REFERENCES


