HOMOGENEOUS SOLUTIONS FOR A CLASS OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS

ABDULLAH ALTIN* and EUTQUIO C. YOUNG**

* Faculty of Sciences, University of Ankara, Beştevler, Ankara, TURKEY
** Florida State University, Tallahassee, Florida, USA

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1. INTRODUCTION

We recall that a spherical harmonic is a homogeneous function of $x$, $y$, $z$ of certain degree $n$ which satisfies Laplace equation. Thus, if $V(x,y,z)$ is such a function of degree $\lambda$, then $xV_x + yV_y + zV_z = \lambda V(x,y,z)$, and $\Delta V = V_{xx} + V_{yy} + V_{zz} = 0$. An important result in the theory of harmonic functions is that any harmonic function can be expressed in a series involving the spherical harmonics.

In this paper we shall study homogeneous functions which satisfy the general elliptic-ultrahyperbolic partial differential equation

$$L(u) = \sum_{i=1}^{n} \left( \frac{\partial^2 u}{\partial x_i^2} + \frac{\alpha_i}{x_i} \frac{\partial u}{\partial x_i} \right) \pm \sum_{j=1}^{s} \left( \frac{\partial^2 u}{\partial y_j^2} + \frac{\beta_j}{y_j} \frac{\partial u}{\partial y_j} \right) + \frac{\gamma}{r^2} u = 0$$

(1)

where $\alpha_i$ (1≤i≤n), $\beta_j$ (1≤j≤s) and $\gamma$ are real parameters and

$$r^2 = \sum_{i=1}^{n} x_i^2 \pm \sum_{j=1}^{s} y_j^2 = |x|^2 \pm |y|^2$$

(2)

The domain of the operator $L$ is the set of all real valued functions $u(x,y)$ of class $C^2(D)$, where $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_s)$ denote points in $\mathbb{R}^n$ and $\mathbb{R}^s$, respectively, and $D$ is a regularity domain of $u$ in $\mathbb{R}^{n+s}$. Clearly the equation (1) includes some of the well known classical equations of mathematical physics such as the Laplace equation, the wave equation and EPD and GASPT equations [1-5]. The equation (1) was considered by Altin [2] for which some expansion formulas for solutions of the iterated forms of the equation were given.
2. HOMOGENEOUS SOLUTIONS

We first give some properties of the operator \( L \). In [2] the following two properties of \( L \) are given.

(i) For any real parameter \( m \),

\[
L(r^m) = [m(m + \phi) + \gamma] r^{m-2}
\]

where

\[
\phi = n + s - 2 + \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{s} \beta_j
\]

(ii) If \( u, v \in C^2(D) \) are any two functions, then the operator \( L \) satisfies the relation

\[
L(uv) = uL(v) + vL(u) - \frac{\gamma}{2} uv + 2 \left( \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \pm \sum_{j=1}^{s} \frac{\partial u}{\partial y_j} \frac{\partial v}{\partial y_j} \right)
\]

In (5), taking \( u = r^m \) and \( v = V_\lambda(x,y) \) which is a homogeneous function of degree \( \lambda \), we then obtain the formula

\[
L(r^m V_\lambda) = m(m + 2\lambda + \phi) r^{m-2} V_\lambda + r^m L(V_\lambda)
\]

This formula will play an important role in finding homogeneous solutions of our equation (1). By making use of the formula (6) we shall prove the following theorem.

**Theorem 1.** Let \( V_\lambda(x,y) \in C^n(D) \) be any homogeneous function of degree \( \lambda \). If \( 2\lambda + \phi \) is not a positive even number, then the function

\[
W_\lambda(x,y) = \left\{ 1 + \sum_{q=1}^{\infty} \left( -1 \right)^q a_q(\lambda,\phi) r^{2q} L^q \right\} V_\lambda(x,y)
\]

where

\[
a_q(\lambda,\phi) = \frac{1}{2.4...2q(2\lambda+\phi-2)(2\lambda+\phi-4)...(2\lambda+\phi-2q)}
\]

and

\[
L^{q+1} = L(L^q) \text{ for } q = 1,2,...
\]

is a homogeneous solution of degree \( \lambda \) of the equation (1).

**Proof.** Using the properties of homogeneous functions and the definition of \( L \), we can see that \( L^q(V_\lambda(x,y)) \) is a homogeneous function of
degree \( \lambda - 2q \) for any positive integer \( q \). Since the factor \( r^{2q} \) is homogeneous of degree \( 2q \), each term \( r^{2q} L^q(V_\lambda) \) of (7) is again a homogeneous function of degree \( \lambda \), and therefore the limit function \( W_\lambda(x,y) \) will be also a homogeneous function of the same degree \( \lambda \). Hence, by the relation (6) we have

\[
L[r^{2q} L^q(V_\lambda)] = 2q[2q + 2(\lambda - 2q) + \phi]r^{2q - 2} L^q(V_\lambda) + r^{2q} L^{q+1}(V_\lambda) \quad (9)
\]

\[
= 2q(2\lambda + \phi - 2q)r^{2q - 2} L^q(V_\lambda) + r^{2q} L^{q+1}(V_\lambda)
\]

Now let us apply the operator \( L \) on both sides of (7) and use the formula (9). We obtain

\[
L(W_\lambda) = L(V_\lambda) + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda,\phi) r^{2q} L[r^{2q} L^q(V_\lambda)]
\]

\[
= L(V_\lambda) + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda,\phi) \left\{ (2q)(2\lambda + \phi - 2q) r^{2q - 2} L^q(V_\lambda) + r^{2q} L^{q+1}(V_\lambda) \right\}
\]

\[
= L(V_\lambda) - a_1(\lambda,\phi) 2(2\lambda + \phi - 2) L(V_\lambda)
\]

\[
+ \sum_{q=2}^{\infty} (-1)^q \left\{ (2q)(2\lambda + \phi - 2q) a_q(\lambda,\phi) - a_{q-1}(\lambda,\phi) \right\} r^{2q} L^q(V_\lambda).
\]

On the other hand from the definition of \( a_q(\lambda,\phi) \), it is clear that

\[
2(2\lambda + \phi - 2)a_1(\lambda,\phi) = 1
\]

and

\[
2q(2\lambda + \phi - 2q)a_q(\lambda,\phi) = a_{q-1}(\lambda,\phi) \quad ; \quad q = 2, 3, ...
\]

Therefore, \( L(W_\lambda) = 0 \), which proves our theorem.

3. SOLUTIONS FOR THE ITERATED EQUATION \( L^p u = 0 \).

First we shall prove the following lemma.

**Lemma 1.** Let \( V_\lambda(x,y) \) be any homogeneous function of degree \( \lambda \). Then for any positive integer \( p \) and for any real number \( m \)

\[
L^p(r^m V_\lambda) = \sum_{k=0}^{p} c(p,k) r^{m-2k} L^p(V_\lambda) \quad (10)
\]

where

\[
L^0(V_\lambda) = V_\lambda, \quad c(0,0) = c(p,0) = 1, \quad c(p,1) = mp(m+2-2p+2\lambda+\phi),
\]
\[ c(p,k) = c(p-1,k)+(m+2-2k)(m+2-4p+2k+2\lambda+\phi)c(p-1,k-1); \quad k = 1,\ldots,p-1 \]
\[ c(p,p) = \prod_{j=0}^{p-1} (m-2j)(m-2j+2\lambda+\phi) \text{ and } c(p,k) = 0 \text{ for } k > p. \]

**Proof.** Applying the operator \( L \) on both sides of the formula (6) and noting that \( L(V_{\lambda}) \) is a homogeneous function of degree \( \lambda-2 \), we have
\[
L^2(r^mV_{\lambda}) = m(m+2\lambda+\phi)((m-2)(m-2+2\lambda+\phi)r^{m-4}V_{\lambda}+r^{m-2}L(V_{\lambda})) \\
= m^2L^2(V_{\lambda})+2m(m-2\lambda+\phi)r^{m-2}L(V_{\lambda}) \\
+ m(m-2)(m+2\lambda+\phi)(m-2\lambda+\phi)r^{m-4}V_{\lambda} \\
= c(2,0)r^{m-4}L^2(V_{\lambda})+c(2,1)r^{m-2}L(V_{\lambda})+c(2,2)r^{m-4}V_{\lambda}
\]
Hence by induction we obtain the formula (10). We note that if \( V_{\lambda} \) is a solution of the equation \( L(u) = 0 \), then our formula (10) takes the form
\[
L^p(r^mV_{\lambda}) = c(p,p)r^{m-2p}V_{\lambda} \\
= r^{m-2p} \prod_{j=0}^{p-1} (m-2j)(m-2j+2\lambda+\phi)V_{\lambda} \quad (11)
\]

By making use of Lemma 1 we shall now establish the following theorem.

**Theorem 2.** Let \( V_{\lambda_j}(x,y) \) be any \( p \) homogeneous integral functions of degree \( \lambda_j \) for \( j = 0,1,\ldots,p-1 \), respectively. Then the functions
\[
(a) \quad u_1 = \sum_{j=0}^{p-1} r^{\lambda_j} \left\{ 1 + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda_j,\phi)r^{2q}L_q \right\} V_{\lambda_j}(x,y) \\
(b) \quad u_2 = \sum_{j=0}^{p-1} r^{\lambda_j+\phi} \left\{ 1 + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda_j,\phi)r^{2q}L_q \right\} V_{\lambda_j}(x,y)
\]
satisfy the iterated equation \( L^p(u) = 0 \). Here \( L, r, \phi \) and \( a_q(\lambda,\phi) \) are defined by (1), (2), (4) and (8) respectively.

**Proof.** Since \( V_{\lambda_j} \) is a homogeneous integral function of degree \( \lambda_j \), by Theorem 1
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\[ W_{\lambda_j}(x,y) = \left\{ 1 + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda_j, \phi) r^{2q} L^q \right\} V_{\lambda_j}(x,y) \]

is a homogeneous solution of degree \( \lambda_j \) of the equation \( L(u) = 0 \).

Therefore, from the formula (11) of Lemma 1, we have

\[ L^p(r^m W_{\lambda_j}) = r^{m-2p} \prod_{j=0}^{p-1} (m-2j)(m-2j+2\lambda_j+\phi) W_{\lambda_j} \]  \hspace{1cm} (12)

Thus, by (12), for \( j = 0,1,...,p-1 \), we have

\[ L^p[r^{2j} W_{\lambda_j}] = 0 \quad \text{and} \quad L^p[r^{2j-2\lambda_j-\phi} W_{\lambda_j}] = 0 \]

Hence, by the principle of superposition, it follows that \( u_1 \) and \( u_2 \) both satisfy the equation \( L^p(u) = 0 \).

We notice that the solution \( u_1 \) is a special case of Almansi's expansion for the equation (1) and the solution \( u_2 \) is a homogeneous function expansion for the same equation (1). Both of them were obtained in [2] using a different method.

4. SOME REMARKS

(i) Suppose \( V_{\lambda} \) is a homogeneous integral function of degree \( \lambda \) such that \( 2\lambda+\phi \) is not a positive even number. Since the function

\[ W_{\lambda}(x,y) = \left\{ 1 + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda, \phi) r^{2q} L^q \right\} V_{\lambda}(x,y) \]

is a solution of the equation (1) and since Kelvin principle is valid for the same equation [2,3], the function

\[ u(x,y) = r^{\phi} W_{\lambda}(\xi, \eta) \]

is also a solution of the same equation (1). Here \( \xi = (\xi_1, ..., \xi_n) \), \( \eta = (\eta_1, ..., \eta_n) \) and \( \xi_i = x_i/r^2 \), \( 1 \leq i \leq n \), \( \eta_j = y_j/r^2 \), \( 1 \leq j \leq s \) and \( r \) and \( \phi \) are defined before by (2) and (4).

(ii) In [2] it was shown that if \( V_{\lambda}(x,y) \) is a homogeneous solution of degree \( \lambda \) of the equation (1), then

\[ L^p[r^m V_{\lambda}(\xi, \eta)] = r^{m-2p} \prod_{j=0}^{p-1} (m-2j+\phi)(m-2j-2\lambda) V_{\lambda}(\xi, \eta) \]  \hspace{1cm} (13)
Using Theorem 2 and the formula (13), we can give two more solution for the iterated equation \( L^p u = 0 \).

Let \( V_{\lambda_j}(x,y) \) be any \( p \) homogeneous integral function of degree \( \lambda_j \) for \( j = 0, 1, ..., p-1 \) and define \( W_{\lambda_j}(x,y) \) as

\[
W_{\lambda_j}(x,y) = \left\{ 1 + \sum_{q=1}^{\infty} (-1)^q a_q (\lambda_j + \phi) r^{2q} L^q \right\} V_{\lambda_j}(x,y), \ j = 0, ..., p-1
\]

which are homogeneous solution of degree \( \lambda_j \) of the equation (1). From (13) we can say that

\[
u_3(x,y) = \sum_{j=0}^{p-1} r^{2j-\phi} W_{\lambda_j}(\xi, \eta)
\]

and

\[
u_4(x,y) = \sum_{j=0}^{p-1} r^{2j+\lambda_j} W_{\lambda_j}(\xi, \eta)
\]

are also solutions of the iterated equation \( L^p(u) = 0 \).

(iii) It is clear that by a simple linear transformation, Theorem 1 can be readily extended to the more general equation of the form

\[
L_1(u) = \sum_{i=1}^{n} a_i \frac{\partial^2 u}{\partial i^2} + \alpha_i \frac{\partial u}{\partial t_i} \pm \sum_{j=1}^{s} \left( b_j \frac{\partial u}{\partial z_j^0} + \beta_j \frac{\partial u}{\partial z_j} \right) + \gamma u = 0
\]

(14)

where \( a_i, b_j, \alpha_i, \beta_j \) are real parameters \( (a_i \neq 0, b_j \neq 0) \), \( t^0 = (t_1^0, ..., t^n) \) and \( z^0 = (z_1^0, ..., z_s^0) \) are fixed points in \( \mathbb{R}^n \) and \( \mathbb{R}^s \), respectively, and \( r_1 \) denoted by

\[
r_1^2 = \sum_{i=1}^{n} \left( \frac{t_i^0}{a_i} \right)^2 \pm \sum_{j=1}^{s} \left( \frac{z_j^0}{b_j} \right)^2
\]

REFERENCES


