QUASI-HADAMARD PRODUCT OF p-VALENT FUNCTIONS

M.K. AOUF, A. SHAMANDY and M.F. YASSEN

Department of Mathematics Faculty of Science University of Mansoura Mansoura, EGYPT.

(Received May 3, 1994; Accepted April 25, 1995)

ABSTRACT

The authors establish certain results concerning the quasi-Hadamard product of analytic and p-valent functions with negative coefficients analogous to the results due to Vinod Kumar.

1. INTRODUCTION

Let \( S_p(z, \beta, \lambda) \) denote the class of functions of the form

\[
f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, \ldots\})
\]

(1.1)

which are analytic and p-valent in the unit disc \( U = \{z: |z| < 1\} \) and satisfy the condition

\[
\left| \frac{zf'(z)}{f(z)} - p \right| < \beta \quad \left| \frac{zf'(z)}{f(z)} + p - \lambda (z+1) \right| < \beta
\]

(1.2)

for some \( \alpha \) \((0 \leq \alpha \leq 1)\), \( \beta \) \((0 < \beta \leq 1)\), \( \lambda \) \((0 \leq \lambda < p)\) and for all \( z \in U \). The class \( S_p(z, \beta, \lambda) \) was studied by Owa and Aouf [4].

Throughout the paper, let the functions of the form

\[
f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_p > 0, \ a_{p+n} \geq 0, \ p \in \mathbb{N}), \quad \text{(1.3)}
\]

\[
f_{1}(z) = a_{p,1} z^p - \sum_{n=1}^{\infty} a_{p+n,1} z^{p+n} \quad (a_{p,1} > 0, \ a_{p+n,1} \geq 0, \ p \in \mathbb{N}), \quad \text{(1.4)}
\]
\[ g(z) = b_p z^p - \sum_{n=1}^{\infty} b_{p+n} z^{n+p} \quad (b_p > 0, b_{p+n} \geq 0, p \in \mathbb{N}), \]  
\[ g_3(z) = b_{p_3} z^{p_3} - \sum_{n=1}^{\infty} b_{p+n_3} z^{n+p} \quad (b_{p_3} > 0, b_{p+n_3} \geq 0, p \in \mathbb{N}), \]  
be analytic and \( p \)-valent in \( U \).

Let \( S_p^* (x, \beta, \lambda) \) denote the class of functions \( f(z) \) of the form (1.3) and satisfying (1.2) for some \( x, \beta, \lambda \) and for all \( z \in U \). Also let \( C_p^* (x, \beta, \lambda) \) denote the class of functions of the form (1.3) such that

\[ \frac{zf'(z)}{p} \in S_p^* (x, \beta, \lambda). \]

We note that when \( a_p = x = \beta = 1 \), the classes \( S_p^* (1, 1, \lambda) = T_p^* (p, \lambda) \) and \( C_p^* (1, 1, \lambda) = C (p, \lambda) \) were studied by Owa [3].

Using similar arguments as given by Owa [3] we can easily prove the following analogous results for functions in the classes \( S_p^* (x, \beta, \lambda) \) and \( C_p^* (x, \beta, \lambda) \).

A function \( f(z) \) defined by (1.3) belongs to the class \( S_p^* (x, \beta, \lambda) \) if and only if

\[ \sum_{n=1}^{\infty} \left[ \frac{n (1 + x \beta) + \beta (1 + x) (p-\lambda)}{a_{p+n}} \right] \leq \beta (1 + x) (p-\lambda) a_p \]  
(1.7)

and \( f(z) \) defined by (1.3) belongs to the class \( C_p^* (x, \beta, \lambda) \) if and only if

\[ \sum_{n=1}^{\infty} \left[ \frac{\binom{p+n}{p}}{p} \right] \frac{n (1 + x \beta) + \beta (1 + x) (p-\lambda)}{a_{p+n}} \leq \beta (1 + x) (p-\lambda) a_p. \]  
(1.8)

We now introduce the following class of analytic and \( p \)-valent functions which plays an important role in the discussion that follows:

A function \( f(z) \), defined by (1.3), belongs to the class \( S_p^* (x, \beta, \lambda) \) if and only if
\[
\sum_{n=1}^{\infty} \left( \frac{p+n}{p} \right)^k \{n (1 + \alpha \beta) + \beta (1 + \alpha) (p-\lambda)\} a_{p+n} \leq \beta (1 + \alpha) (p-\lambda) a_p,
\]

(1.9)

where \(0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \lambda < p\) and \(k\) is any fixed nonnegative real number.

We note that, for every nonnegative real number \(k\), the class \(S_{p, k}^* (\alpha, \beta, \lambda)\) is nonempty as the functions of the form

\[
f(z) = a_p z^p - \sum_{n=1}^{\infty} \frac{\beta (1 + \alpha) (p-\lambda) a_p}{\left( \frac{p+n}{p} \right)^k \{n (1 + \alpha \beta) + \beta (1 + \alpha)(p-\lambda)\}} \lambda_{p+n} z^{p+n},
\]

(1.10)

where \(a_p > 0, \lambda_{p+n} \geq 0\) and \(\sum_{n=1}^{\infty} \lambda_{p+n} \leq 1\), satisfy the inequality (1.9).

It is evident that \(S_{p, 1}^* (\alpha, \beta, \lambda) = C_p^* (\alpha, \beta, \lambda)\) and, for \(k=0, S_{p, k}^* (\alpha, \beta, \lambda)\) is identical to \(S_{p}^* (\alpha, \beta, \lambda)\). Further, \(S_{p, k}^* (\alpha, \beta, \lambda) \subset S_{p, k-1}^* (\alpha, \beta, \lambda) \subset \ldots \subset S_{p, 2}^* (\alpha, \beta, \lambda) \subset C_p^* (\alpha, \beta, \lambda) \subset S_{p}^* (\alpha, \beta, \lambda)\).

Let us define the quasi-Hadamard product of the functions \(f(z)\) and \(g(z)\) by

\[
f_g (z) = a_p b_p z^p - \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}.
\]

(1.11)

Similarly, we can define the quasi-Hadamard product of more then two functions.

In this paper we establish certain results concerning the quasi-Hadamard product of functions in the classes \(S_{p, k}^* (\alpha, \beta, \lambda), S_{p}^* (\alpha, \beta, \lambda)\) and \(C_p^* (\alpha, \beta, \lambda)\) analogous to the results due to Vinod Kumar [1, 2].

2. THE MAIN THEOREMS

Theorem 1. Let functions \(f_i(z)\) defined by (1.4) be in the class \(C_p^* (\alpha, \beta, \lambda)\) for every \(i = 1, 2, \ldots, m\); and let the functions \(g_j(z)\)
defined by (1.6) be in the class $S^*_p (z, \beta, \lambda)$ for every $j = 1, 2, \ldots, q$. Then, the quasi–Hadamard product $f_1 \ast f_2 \ast \ldots \ast f_m \ast g_1 \ast g_2 \ast \ldots \ast g_q(z)$ belongs to the class $S^*_p; 2m+q-1 (z, \beta, \lambda)$.

**Proof:** We denote the quasi–Hadamard product $f_1 \ast f_2 \ast \ldots \ast f_m \ast g_1 \ast g_2 \ast \ldots \ast g_q(z)$ by the function $g(z)$, for the sake of convenience. Clearly,

$$h(z) = \left\{ \prod_{i=1}^{m} a_{p, i} \prod_{j=1}^{q} b_{p, j} \left\{ \frac{\beta}{n} \sum_{n=1}^{\infty} \prod_{i=1}^{m} a_{p+n,i} \prod_{j=1}^{q} b_{p+n,j} \right\} \right\} \left\{ z^{p+n} \right\} \ (2.1)$$

To prove the theorem, we need to show that

$$\sum_{n=1}^{\infty} \left( \frac{p+n}{p} \right)^{2m+q-1} \left\{ n (1+\beta) + \beta (1+\alpha) (p-\lambda) \right\} \prod_{i=1}^{m} a_{p+n, i} \prod_{j=1}^{q} b_{p+n, j} \right\} \leq \beta (1+\alpha) (p-\lambda) \left( \prod_{i=1}^{m} a_{p, i} \prod_{j=1}^{q} b_{p, j} \right). \ (2.2)$$

Since $f_1(z) \in C^*_p (z, \beta, \lambda)$, we have

$$\sum_{n=1}^{\infty} \left( \frac{p+n}{p} \right)^{\left\{ n (1+\beta) + \beta (1+\alpha) (p-\lambda) \right\}} \prod_{i=1}^{m} a_{p+n, i} \leq \beta (1+\alpha) (p-\lambda) \prod_{i=1}^{m} a_{p, i}, \ (2.3)$$

for every $i = 1, 2, \ldots, m$. Therefore

$$\left( \frac{p+n}{p} \right)^{\left\{ n (1+\beta) + \beta (1+\alpha) (p-\lambda) \right\}} \prod_{i=1}^{m} a_{p+n, i} \leq \beta (1+\alpha) (p-\lambda) \prod_{i=1}^{m} a_{p, i}$$

or

$$a_{p+n, i} \leq \left[ \frac{\beta (1+\alpha) (p-\lambda)}{\left( \frac{p+n}{p} \right)^{\left\{ n (1+\beta) + \beta (1+\alpha) (p-\lambda) \right\}}} \prod_{i=1}^{m} a_{p, i} \right]$$

for every $i = 1, 2, \ldots, m$. The right–hand expression of this last inequality is not greater than $\left( \frac{p+n}{p} \right)^{-2} a_{p, i}$. Hence
for every $i = 1, 2, \ldots, m$. Similarly, for $g_j(z) \in S^*_p(\alpha, \beta, \lambda)$, we have

$$\sum_{n=1}^{\infty} \left\{ n \left( 1 + x \beta \right) + \beta \left( 1 + x \right) (p-\lambda) \right\} b_{p_n, j} \leq \beta \left( 1 + x \right) (p-\lambda) b_{p, j} \quad (2.5)$$

for every $j = 1, 2, \ldots, q$. Whence we obtain

$$b_{p_n, j} \leq \left( \frac{p+n}{p} \right)^{-1} b_{p, j} \quad (2.6)$$

for every $j = 1, 2, \ldots, q$.

Using (2.4) for $i = 1, 2, \ldots, m$, (2.6) for $j = 1, 2, \ldots, q-1$, and (2.5) for $j = q$, we get

$$\sum_{n=1}^{\infty} \left[ \left( \frac{p+n}{p} \right)^{2m+q-1} \left\{ n \left( 1 + x \beta \right) + \beta \left( 1 + x \right) (p-\lambda) \right\} \prod_{i=1}^{m} a_{p_n, i} \prod_{j=1}^{q} b_{p_n, j} \right]$$

$$\leq \sum_{n=1}^{\infty} \left[ \left( \frac{p+n}{p} \right)^{2m+q} \left\{ n \left( 1 + x \beta \right) + \beta \left( 1 + x \right) (p-\lambda) \right\} b_{p_n, q} \right.$$

$$\left. \cdot \left\{ \left( \frac{p+n}{p} \right)^{-2m} \left( \frac{p+n}{p} \right)^{-(q-1)} \prod_{i=1}^{m} a_{p, i} \prod_{j=1}^{q-1} b_{p, j} \right\} \right]$$

$$= \sum_{n=1}^{\infty} \left[ \left\{ n \left( 1 + x \beta \right) + \beta (1 + x) (p-\lambda) \right\} b_{p_n, q} \right] \left( \prod_{i=1}^{m} a_{p, i} \prod_{j=1}^{q-1} b_{p, j} \right)$$

$$\leq \beta \left( 1 + x \right) (p-\lambda) \left( \prod_{i=1}^{m} a_{p, i} \prod_{j=1}^{q} b_{p, j} \right).$$

Hence $h(z) \in S^*_p, 2m+q-1 (x, \beta, \lambda)$. This completes the proof.

We note that the required estimate can also be obtained by using (2.4) for $i = 1, 2, \ldots, m-1$, (2.6) for $j = 1, 2, \ldots, q$, and (2.3) for $i=m$.

Now we discuss the applications of Theorem 1. Taking into account the quasi-Hadamard product of the functions $f_1(z), f_2(z), \ldots, f_m(z)$ only, in the proof of Theorem 1, and using (2.4) for $i = 1, 2, \ldots, m-1$ and (2.3) for $i=m$, we are led to
Corollary 1. Let the functions $f_i(z)$ defined by (1.4) belong to the class $C^*_{p_1}(z, \beta, \lambda)$ for every $i = 1, 2, \ldots, m$. Then the quasi-Hadamard product $f_1 f_2 \ldots f_m(z)$ belongs to the class $S^*_{p_2 2m-1}(z, \beta, \lambda)$.

Next, taking into account the quasi-Hadamard product of the functions $g_1(z), g_2(z), \ldots, g_q(z)$ only, in the proof of Theorem 1, and using (2.6) for $j = 1, 2, \ldots, q-1$ and (2.5) for $j = q$, we are led to.

Corollary 2. Let the functions $g_j(z)$ defined by (1.6) belong to the class $S^*_{p_1}(z, \beta, \lambda)$ for every $j = 1, 2, \ldots, q$. Then, the quasi-Hadamard product $g_1 g_2 \ldots g_q(z)$ belongs to the class $S^*_{p_2 q-1}(z, \beta, \lambda)$.

REFERENCES


