GEOMETRICAL APPROACH TO BALANCED
INCOMPLETE BLOCK DESIGNS

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ABSTRACT

Construction and analysis of balanced incomplete block design needs special care. There are several methods to construct balanced incomplete block designs. We tried to give relations between parameters of balanced incomplete block design and properties of projective geometry and related geometries. We offered some geometrical examples. Some of them can be considered original.

1. INTRODUCTION

There are several methods to construct BIB designs; some of them are Method of Symmetrically Repeated Difference, Method of Mutually Orthogonal Latin Squares, Method of Near–Rings, Hadamard Matrices Method [3].

GF(p^n) has been used to obtain all BIB designs. Using finite analytical geometries, the construction can be done easily. For example, we can obtain at most p^n–1 orthogonal Latin Squares in p elements. Since $\text{EG}(2, p^n)$ is a kind of Affine planar, one can obtain $\text{PG}(2, p^n)$ by agumentation of Affine planar. $\text{EG}(2, p^n)$ and $\text{PG}(2, p^n)$ will represent several BIB designs.

Definition 1.1: Consider two sets T and B with their elements beings treatments and blocks, respectively. There are t treatments and b blocks in BIB design that satisfy the following conditions:

1– Each block has exactly k members,
2– Each treatment occurs in exactly r blocks,
3– Every pair of treatments occur in exactly $\lambda$ blocks.
The parameters of BIB design satisfy the following equations
1) \( n = bk = rt \),
2) \( \lambda = r(k-1)/(t-1) \).

Where \( n \) is the total number of observations.

Definition 1.2: BIB design is said to be symmetrical if \( t = b \).

II. FINITE PROJECTIVE GEOMETRY AND FINITE EUCLID GEOMETRY

There is GF\((p^n)\) of order \( p^n \) to every prime number \( p \) and every positive integer \( n \). A point in the \( m \) dimensional finite projective geometry PG\((m, p^n)\) is an ordered set of \((m + 1)\) elements of GF\((p^n)\), not all of which are equal to 0.

Let \((x_1, x_2, \ldots, x_{m+1})\) and \((x'_1, x'_2, \ldots, x'_{m+1})\) are two sets. If

\[
\gamma_i = \gamma x'_i, \quad i = 1, 2, \ldots, m + 1, \quad \gamma \neq 0, \quad \gamma \in \text{GF}(p^n) \quad (2.1)
\]

two sets represents the same point. Let \( P_1 = (x_1, x_2, \ldots, x_{m+1}) \) and \( P_2 = (x'_1, x'_2, \ldots, x'_{m+1}) \) be two distinct points. We define the line:

\[
\gamma_1 P_1 + \gamma_2 P_2 = \gamma_1 x_1 + \gamma_2 x'_1 + \gamma_1 x_2 + \gamma_2 x'_2 + \ldots + \gamma_1 x_{m+1} + \gamma_2 x'_{m+1} \quad (2.2)
\]

where \( \gamma_1, \gamma_2 \in \text{GF}(p^n) \) and at least one of the \( \gamma_1 \neq 0 \).

The system of points and lines which are defined above is called analytic projective geometry of GF\((p^n)\) of \( m \) dimensions.

Number of different points in PG\((m, p^n)\) are given by

\[
\frac{p^{n(m+1)} - 1}{p^n - 1} = 1 + p^n + \ldots + p^{mn}. \quad (2.3)
\]

Each subset of PG\((m, p^n)\) of \( u \) dimensions is again PG\((u, p^n)\) and includes \( 1 + p^n + \ldots + p^{un} \) points. Each PG\((u, p^n)\) is a set of \((u + 1)\) linearly independent points. \( P_3 \) can be constituted by \((1 + p^n + \ldots + p^{mn})\) different selections. Similarly \( P_2 \) can be constituted by \( p^n + \ldots + p^{mn} \) and \( P_3 \) can be constituted by \( p^{2n} + \ldots + p^{mn} \) different selections. After choosing \( t \) points, we can take \((t + 1)\) points outside PG \((t-1, p^n)\), provided \( t < u + 1 \). We can find the number of ordered sets of \((u + 1)\) independent points in PG\((m, p^n)\). This number is

\[
(1 + p^n + \ldots + p^{mn}) \ldots (p^{un} + \ldots + p^{mn}).
\]

Similarly number of ordered sets of \((u + 1)\) independent points in PG\((u, p^n)\) is
(1 + p^n + \ldots + p^{un}) \ldots (p^{(u-1)n} + p^{un})p^{un}.

So number of PG(u, p^n) in PG(m, p^n) is

\[
\frac{(1 + p^n + \ldots + p^{mn}) \ldots (p^{un} + \ldots + p^{mn})}{(1 + p^n + \ldots + p^{un}) \ldots (p^{(u-1)n} + p^{un}) p^{un}}. \tag{2.4}
\]

Number of PG(s, p^n) in PG(m, p^n) that include a given PG(u, p^n) is

\[
\frac{(p^{(u+1)n} + \ldots + p^{mn}) \ldots (p^{sn} + \ldots + p^{mn})}{(p^{(u+1)n} + \ldots + p^{sn}) \ldots + (p^{(s-1)n} + p^{sn}) p^{sn}}. \tag{2.5}
\]

We can regard blocks and treatments of BIB design as the lines and the points of PG(m, p^n).

**Theorem 1.** The subset PG(s, p^n) of PG(m, p^n) constitutes a BIB design with the following parameters [2]:

\[
b (s, m, p^n) = \frac{(1 + p^n + \ldots + p^{mn}) \ldots (p^{sn} + \ldots + p^{mn})}{(1 + \ldots + p^{sn}) \ldots (p^{(s-1)n} + p^{sn}) p^{sn}},
\]

\[
t (m, p^n) = 1 + p^n + \ldots + p^{mn},
\]

\[
k (s, p^n) = 1 + p^n + \ldots + p^{sn},
\]

\[
r (s, m, p^n) = \frac{(p^n + \ldots + p^{mn}) \ldots (p^{sn} + \ldots + p^{mn})}{(p^n + \ldots + p^{sn}) \ldots (p^{(s-1)n} + p^{sn}) p^{sn}}, \tag{2.6}
\]

\[
\lambda (s, m, p^n) = \begin{cases} 
1 & \text{for } s = 1 \\
\frac{(p^{2n} + \ldots + p^{mn}) \ldots (p^{3n} + \ldots + p^{mn})}{(p^{2n} + \ldots + p^{mn}) \ldots (p^{(s-1)n} + p^{sn})p^{sa}} & \text{for } s > 1.
\end{cases}
\]

If we delete any PG(m−1, p^n) from a PG(m, p^n), we can obtain EG(m, p^n).

**Theorem 2:** The subspace EG(s, p^n) of EG(m, p^n) constitute a BIB design with parameters [2]:

\[
b = b (s, m, p^n) - b (s, m-1, p^n)
\]

\[
t = p^{mn},
\]

\[
k = p^{sn},
\]

\[
r = r (s, m, p^n),
\]

\[
\lambda = \lambda (s, m, p^n).
\]

**Theorem 3.** (Bruck-Rayser Theorem): There does not exist a projective plane of order p^n if p^n = 1 (mod 4) or p^n = 2 (mod 4) and p^n is not a sum of two nonnegative integers [1].
According to the Bruck–Ryser theorem, projective planes of some order do not exist.

EXAMPLES

1. For \( b = t = 4, \ r = k = 3, \ \lambda = 2 \), BIB design can be shown as in Table 1.

<table>
<thead>
<tr>
<th>Treatments</th>
<th>BLOCKS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>A</td>
<td>X</td>
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<tr>
<td>B</td>
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<td>C</td>
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Table 1. BIB design for \( b = t = 4, \ r = k = 3, \ \lambda = 2 \)

According to the Bruck–Ryser theorem, the BIB design above can not be a projective plane but can be shown as in Figure 1.

2. For \( t = b = 13, \ k = r = 4, \ \lambda = 1 \) BIB design can be shown as in Table 2.
Table 2. BIB design for t = b = 13, k = r = 4, λ = 1

<table>
<thead>
<tr>
<th>Treatments</th>
<th>BLOCKS</th>
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<tbody>
<tr>
<td>1</td>
<td>X X X X</td>
</tr>
<tr>
<td>2</td>
<td>X X X X</td>
</tr>
<tr>
<td>3</td>
<td>X X X X</td>
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<tr>
<td>4</td>
<td>X X X X</td>
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<td>5</td>
<td>X X X X</td>
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<td>6</td>
<td>X X X</td>
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<td>7</td>
<td>X X X</td>
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<tr>
<td>8</td>
<td>X X X</td>
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<tr>
<td>9</td>
<td>X X X</td>
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<tr>
<td>10</td>
<td>X X X</td>
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<tr>
<td>11</td>
<td>X X X</td>
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<tr>
<td>12</td>
<td>X X X</td>
</tr>
<tr>
<td>13</td>
<td>X X X</td>
</tr>
</tbody>
</table>

Using Theorem 1, one can obtain the BIB design as in Figure 2, which can be represented as PG (2, 3). The points and the lines of PG (2, 3) are:

Points
P₁: 100 P₅: 101 P₉: 10–1 P₁₃: 0–11
P₂: 010 P₆: 011 P₁₀: -111
P₃: 001 P₇: 111 P₁₁: 1–11
P₄: 110 P₈: 1–10 P₁₂: 11–1

Lines
[001]: P₁ P₂ P₄ P₈
[010]: P₁ P₃ P₅ P₉
[0–11]: P₁ P₆ P₇ P₁₀
[011]: P₁ P₁₁ P₁₂ P₁₃
[100]: P₂ P₃ P₆ P₁₃
[10-1] :  $P_2 \quad P_5 \quad P_7 \quad P_{11}$
[101] :  $P_2 \quad P_9 \quad P_{10} \quad P_{12}$
[1-10] :  $P_3 \quad P_4 \quad P_7 \quad P_{12}$
[110] :  $P_3 \quad P_8 \quad P_{10} \quad P_{11}$
[-111] :  $P_4 \quad P_5 \quad P_{10} \quad P_{13}$
[1-11] :  $P_4 \quad P_6 \quad P_9 \quad P_{11}$
[11-1] :  $P_5 \quad P_6 \quad P_8 \quad P_{12}$
[111] :  $P_7 \quad P_8 \quad P_9 \quad P_{13}$

The BIB design is a projective plane which is shown in Figure 2.
Let's delete the line [001] and all its points from PG (2, 3). Using Theorem 2, one can obtain the BIB design as in Table 3 which can be represented as EG (2, 3).

3. For $t = 9$, $b = 12$, $r = k = 4$, $\lambda = 1$, BIB design can be shown as in Table 3.

<table>
<thead>
<tr>
<th>Treatments</th>
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<tbody>
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<td>2</td>
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<tr>
<td>12</td>
<td>X</td>
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<tr>
<td>13</td>
<td>X</td>
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The BIB design which is related to Table 3 is a EG (2,3) which is shown in Figure 3.

![Figure 3]
The designs in above examples are some of the well known designs in statistics and employed in experimental work.

ÖZET

Bu çalışmada projektif ve Euclid geometrinin özelliklerinden faydalanarak dengeli tamamlanmamış blok deney düzeninin geometrik gösterimi veültülmüş. Projektif geometrinin doğruları ve noktalardan sırasıyla dengeli tamamlanmamış blok deney düzeninin blokları ve işlemlerine karşılık gelmektedir.

REFERENCES

